Stanisław Domoradzki

ORCID [0000-0002-6511-0812]
Institute of History, University of Rzeszów (Rzeszów, Poland)
stanislawdomoradzki@gmail.com

Mykhailo Zarichnyi

ORCID [0000-0002-6494-2289]
Institute of Mathematics, University of Rzeszów (Rzeszów, Poland)
Ivan Franko Lviv National University, Department of Mathematics (Lviv, Ukraine)
zarichnyi@yahoo.com

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Reflections on the Importance of the *Leonhardi Euleri Opera Omnia*, Volume IV A (2016) and Volume VIII (2018)

Abstract

The article is devoted to two volumes of Leonhard Euler's correspondence with mathematicians and other scientists.

The first of these volumes (in two parts) is devoted to correspondence with Christian Goldbach. We consider selected topics from this correspondence reflecting various branches of mathematics and demonstrate, where possible, the connection of the ideas and results presented there with modern mathematical research.

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The second of these two volumes contains Euler's correspondence with scholars associated with the University of Halle. These letters, with a small exception, have less mathematical content, but allow to create an impression of academic life at that time.

Keywords: *Euler's correspondence*, *Christian Goldbach*, *Goldbach conjecture, Catalan numbers, divergent series*

Refleksje nad znaczeniem dzieła „Leonhardi Euleri Opera Omnia, tom IV A (2016) i tom VIII (2018)"

Abstrakt

Artykuł dotyczy dwóch tomów korespondencji Leonharda Eulera z matematykami i innymi uczonymi. Pierwszy z tych tomów (w dwóch częściach) poświęcony jest korespondencji z Christianem Goldbachem.

Rozważamy wybrane tematy z tej korespondencji, odzwierciedlające różne gałęzie matematyki i pokazujemy, gdzie to możliwe, powiązanie przedstawionych tam idei i wyników ze współczesnymi badaniami matematycznymi.

Drugi z omawianych tomów zawiera korespondencję Eulera z uczonymi związanymi z Uniwersytetem w Halle. Listy te, z małymi wyjątkami, zawierają mniej treści matematycznych, pozwalają na wyobrażenie ówczesnego życia akademickiego.

Słowa kluczowe: *korespondencja Eulera, Christian Goldbach, hipoteza Goldbacha, liczby Catalana, szeregi rozbieżne*

1. Introduction

Leonhard Euler (1707–1783) is one of the greatest mathematicians in human history. He also worked successfully in other fields. Euler's biography is well described in scientific and popular literature. Euler contributed to all branches of his contemporary mathematics, both pure and applied. He introduced the notation we still use today: for function, for summation, and for the imaginary unit. Euler was the author of 866 scientific publications, most of which are listed in the Eneström

Index. Many mathematical concepts and assertions are named after Euler: Euler diagram, Euler equations (fluid dynamics), Euler–Maclaurin formula, Euler's method, Euler approximations, Eulerian circuit, Euler characteristic, Euler's identity, Euler–Mascheroni constant, Euler–Cauchy equation, Euler–Lagrange equation, Euclid–Euler theorem, Goldbach–Euler theorem, etc. The Riemann zeta function is also sometimes called the Euler–Riemann zeta function, since Euler first introduced it for the real variable. Euler is considered the 'father' of graph theory.

Note that to avoid ambiguity, some mathematical objects introduced by Euler have been attributed to authors who studied these objects after Euler. (An example can be Catalan numbers, which will be discussed later.)

Opera Omnia Leonhard Euler (*Leonhardi Euleri Opera omnia*) is a publishing project aimed to publish all of Euler's scientific writing. Work on the publications lasted a whole century. The volumes are divided into four series:

Series I: Opera mathematica (Mathematics).
Series II: Opera mechanica et astronomica (Mechanics and Astronomy).
Series III: Opera physica, Miscellanea (Physics and Miscellaneous).
Series IVA: Commercium epistolicum (Correspondence).
Series IVB: Manuscripts.

Series IVA consists of 9 volumes and the present article is devoted to two of them. Volume IV „Leonhardi Euleri commercium epistolicum cum Cristiano Goldbach" (eds. Franz Lemmermeyer and Martin Mattmüller) is devoted to the lifelong correspondence between Leonhard Euler and Christian Goldbach. While the name of Euler is known almost in every part of mathematics and not only in mathematics, Christian Goldbach needs to be introduced. The biography of Christian Goldbach is well described in various sources related to Euler, but also in individual publications (see e.g. Yushkevich and Kopelevich 1983, 1994). Goldbach was born on March 18, 1690 in Königsberg, in the family of a pastor. He studied mathematics at the University of Königsberg, although mathematics was not his main subject. Subsequently, he travelled around Europe and met with various scientists, among them Leibniz, with whom he exchanged letters. The trip across Europe stretched over more than 10 years, while Goldbach also met and corresponded with Abraham de Moivre and the Bernoulli family. He is best known for his correspondence with these mathematicians. Due to the lack of mathematical journals in the 17th and 18th centuries, the exchange of scientific letters was an opportunity to communicate scientific results quickly as letters were a form of cooperation, exchange of ideas, etc. It was often a self-report in a given field. Euler greatly appreciated the recipients of his correspondence and he praised the effort of reading the letter and expressing an opinion. Correspondence was also a way to disseminate scientific results.

Some historians of mathematics are inclined to think that it was Goldbach who got Euler interested in the number theory and encouraged his research in this direction in every possible way. At the very first glance, the scientific importance of the Euler–Goldbach correspondence is a priory limited by the fact that Goldbach was not a mathematician by profession and he did not conduct mathematical research. However, Goldbach published several mathematical works, but he saw mathematics more as entertainment rather than work. His important place in the history of mathematics is not due to these publications, but, as noted above, rather to his correspondence, which demonstrated his mathematical intuition and ability to formulate problems which turned out to be both interesting and important. Goldbach died in 1764.

In the present paper, we also briefly discuss Volume VIII of Series IVA, which contains Euler's correspondence with scholars connected to the University of Halle. Here we also note that Denkowski (2020) reviewed volume VII Series IVA of Euler's correspondence with his Swiss compatriots in French.

2. Volume IV of Series IVA

2.1. Structure

The volume consists of two parts. The first of them contains the Introduction in English and the correspondence of Goldbach and Euler in original languages (193 letters written in Latin, French and German). The second part contains translations of all letters into English, with scientific comments, as well as the following indices:

– Synoptic table of the correspondence;
– List of Leonhard Euler's works ordered according to their Eneström numbers, i.e., the numbers in the Eneström (1910) index. Gustaf

Eneström (1852–1923) was a Swedish mathematician and historian of mathematics and was mostly famous because of this index, i.e., the list of Euler's works (books, papers, and letters which were important for the development of mathematics or other sciences) completed in 1910–1913. The index includes 866 items. See (Maligranda 2008) for Eneström's biography and Eneström's index;

- Bibliography presented on pages 1159–1200 contains all the references (in the Introduction, in the letters, and in Editor's remarks);
- Systematic subject index; it contains different areas in mathematics (algebra, number theory, analysis, geometry, and others), physics, astronomy, music theory, and anatomy. Also, some letters concern professional activity, academies, publications, teaching etc.;
- Name index with the names mentioned in the letters, in the Introduction, and in the comments to the letters. Also, some implicitly mentioned persons are included;
- List of abbreviations.

An extensive (about 100 pages long) introduction is devoted to biographical information about Goldbach and his contacts with Euler. Although from a certain point, Goldbach was not actively involved in mathematical research, he belonged to a small circle of people with whom Euler shared his results and who could evaluate them properly. The introduction contains tables that illustrate the chronology and statistics of the Euler–Goldbach correspondence. The first table shows the dates of letters, the author, the number of the letter, and finally the language. It can be seen that the correspondence was started in 1729, and in this year two letters were written: the first one by Euler in October and the second one by Goldbach in December, both in Latin. Naturally, these letters are numbered 1 and 2.

The introduction highlights three periods of intense correspondence between Euler and Goldbach, as well as two periods of relative calm:

1) October 1729 – January 1732, when Goldbach lived in Moscow and Euler in Saint Petersburg. During this initial period, the correspondence covered series, Fermat numbers, logarithms, remainders, integration, continued fractions, and more.

2) February 1732 – June 1741. During this period, Euler and Goldbach both lived in Saint Petersburg and there was no need to write with

the same frequency as before. Nevertheless, 18 letters (numbers 20–37) were written during this nine-year period.

3) July 1741 – July 1756. During this period of time, it took about two weeks for letters to travel from Berlin to Moscow, so a month would pass between a letter and a reply. Considering the number of letters, we can conclude that letters were exchanged with almost the maximum frequency possible.

4) August 1756 – May 1762, with only one letter number 184 in this period of war;

5) June 1762 – November 1964. This final period of correspondence lasted until Goldbach's death. The mathematical component of the correspondence is relatively low.

The Introduction also describes the process of editing the Euler–Goldbach correspondence. First of all, archival sources are described in detail. The multilingual nature of the correspondence caused additional problems during publication.

The Editorial principles contain descriptions of the work on the manuscript, while noting such difficulties as the presence of different forms of handwriting and their expression during publication, the use of capital and small letters, abbreviations, punctuation, the presence of diacritics, errors in the original, incomplete or incorrectly formulated sentences, and finally illegible fragments.

All correspondence between Euler and Goldbach has been translated into English, as neither German nor Latin are accepted languages of science, unlike in the 18th century. The goals of the translation are to avoid anachronisms and jargon, and to achieve syntactic simplicity instead of the complex verbal constructions characteristic of that era. Here it is worth noting that the goals stated have been achieved in this edition.

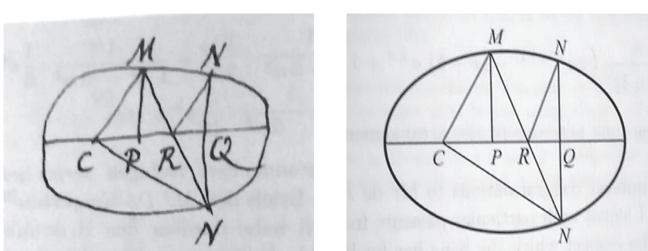

Fig. 1. Figures in the original and the translated letters, in Part 1 and Part 2

The letters often contain drawings which are reproduced in the original texts. At the same time, any stains, etc., have been removed. In the translations, the drawings are made professionally according to relevant standards regarding the straightness, smoothness and uniformity of lines.

2.2. Mathematics in the correspondence

The correspondence between Euler and Goldbach thematically covers almost all sections of pure and applied mathematics contemporary to them. An exception can be the probability theory. It is hardly possible within the framework of a relatively short article to give a complete overview of the results that left an important mark in the history of mathematics. Showing inevitable subjectivism, we will focus here only on a few subjects. One of our tasks is to draw a historical line to modern mathematical research.

2.3. Mathematics in the correspondence: number theory

Most probably, many topics of the correspondence concerned various issues of the number theory. A large part of the letters, about three dozen, were devoted to the problems formulated by Pierre Fermat in his publications. In particular, Euler proved that not all Fermat numbers are prime; he also considered Mersenne numbers.

Recall that Diophantine equations are polynomial equations with integer coefficients, in which the unknowns can take only integer values.

In the correspondence between Euler and Goldbach, various Diophantine equations were considered, in particular:

$$2z^4 \pm 2 = x^2,$$
$$x^2 + y^2 + z^2 - 2x - 2y - 2z + 1 = 0,$$
$$xy(x + y) = a,$$
$$xyz(x + y + z) = a,$$
$$x^2 - ny^2 = 1 \text{ (Pell's equation), etc.}$$

Since Goldbach's name is primarily associated in the history of mathematics with the conjecture that bears his name, we will dwell on it in a little more detail. Goldbach's famous conjecture (every natural number > 2 is the sum of three primes; sometimes it is called the ternary Goldbach conjecture) appears in letter number 51 of June 7, 1742.

In his reply of June 30, 1742 (letter number 52), Euler replies: “That any number which is resolvable into two prime numbers can also be split into as many prime numbers as one wishes, can be illustrated and confirmed from an observation that you, Sir, communicated to me some time ago, when you stated that any even number is the sum of two primes. […] Indeed, I consider the statement that any even number is the sum of two primes to be an utterly certain theorem, notwithstanding the fact that I cannot prove it.”

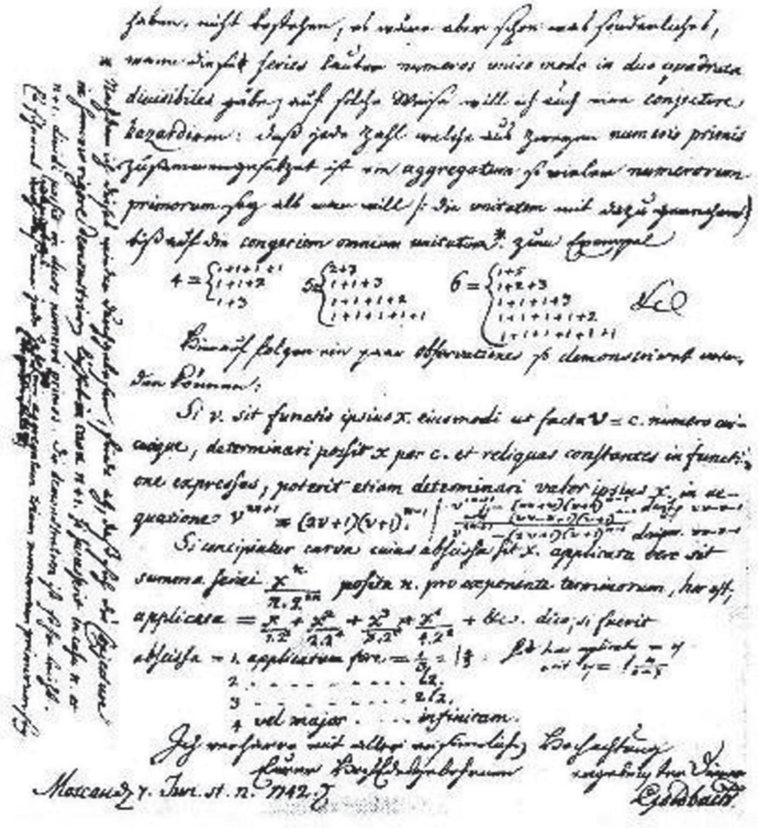

Fig. 2. Letter from Goldbach to Euler of June 7, 1742 (number 51)

In other words, Euler notes that this Goldbach conjecture follows from the following: every even number is the sum of two prime numbers (Goldbach’s binary conjecture).

In 2013, Peruvian mathematician Harald Helfgott published a preprint (Helfgott 2013), where he announced the proof of Goldbach's ternary conjecture. His article was accepted for publication in a mathematical journal, but has not yet been published. The mathematical community accepts Harald Helfgott's proof as correct. See also the preprint (Helfgott 2015) of the book devoted to the solution of Goldbach's ternary conjecture.

In modern mathematics, there are many interesting results in the direction of additive properties of prime numbers. They are related to Goldbach's conjecture, but somewhat weaker. In particular, Lev Schnirelman (1930) proved that there is an effectively computable constant C such that every natural number greater than 1 can be written as the sum of at most C prime numbers.

In 1995, the French mathematician Olivier Ramaré proved that every even number is the sum of at most six primes.

The laureate of the Fields Medal, the American mathematician Terence Tao (2014) proved that any odd number greater than 1 can be written as the sum of no more than five prime numbers. Note that it follows from the ternary Goldbach conjecture that every natural number ≥ 4 is the sum of at most four prime numbers.

However, Goldbach's binary conjecture is still far from being proven. Let us note at the end that due to the simplicity of its formulation, combined with the difficulty to prove or disprove, the (binary) conjecture of Goldbach is repeatedly referenced in pop culture. In particular, it appears in the Spanish movie *Fermat's Room*.

2.4. Mathematics in the correspondence: combinatorics

The term 'combinatorics' does not appear in the subject index of volume IV, and the results concerning the Catalan numbers are present in the Topology section. A history of the Catalan numbers is described in (Pak 2014). The Catalan number C_n is defined as the number of triangulations of convex $(n + 2)$-gon. This notion is introduced in the letter of Euler to Goldbach of September 4, 1751 (no 154). Actually, it can be learned from (Larcombe 1999) that the Catalan numbers have already been known to the Chinese mathematician Ming Antu, but his results became accessible only in 1839, when his book *Quick Methods for Accurate Values of Circle Segments*, written in the 1730s, was published.

In the letter, Euler calculated C_n for $n \leq 10$ by induction and concluded that, in general,

$$C_n = \frac{2 \cdot 6 \cdot 10 \cdot 14 \cdot 18 \cdot 22 \cdots (4n-10)}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdots (n-1)}.$$

Also, Euler notes that

$$1 + 2a + 3a^2 + 14a^3 + 42a^4 + 132a^5 + \ldots = \frac{1 - 2a - \sqrt{1-4a}}{2a^2}.$$

In modern terms, this means that the function on the right-hand side is the generating function for the sequence C_n. The idea of a generating function is explained by Euler in his letter of May 4, 1748 (letter number 127).

In the subsequent letter of October 16, 1751 (number 155), Goldbach remarked that the generating function obtained (he denotes it by A) satisfies the differential equation $1 + aA = A^{1/2}$.

In his letter of December 4, 1951 (number 156), Euler explained how to obtain the formula for the generating function from the expansion

$$\sqrt{1-4a} = 1 - \frac{1}{2} \cdot 4a - \frac{1 \cdot 1}{2 \cdot 4} \cdot 4^2 a^2 - \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6} \cdot 4^3 a^3 - \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8} \cdot 4^4 a^4 - \\ \frac{-1 \cdot 1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10} \cdot 4^5 a^5 - \ldots$$

A few years later, Euler wrote about the product formula to J. A. Segner. Euler's letters have not survived, and Segner's answers are placed in volume VIII (letters number 38 and 39).

The numbers C_n were later called the Catalan numbers in honor of the French-Belgian mathematician Eugène Charles Catalan (1814–1894). To understand why Catalan'sthe Catalan numbers got that name, let us remember that the French mathematician Joseph Liouville (1809–1882) formulated for his students the problem of deriving the product formula from Segner's recursion formula. One of these mathematicians was Catalan, who obtained the now widely known formula

$$C_n = \frac{2n!}{n!(n+1)!} = \binom{2n}{n} - \binom{2n}{n-1} \quad \text{(Catalan 1838).}$$

Actually, Catalan himself used the name the Segner numbers (Catalan 1887). The name Segner–Euler numbers was also used in some old publications. As Pak (2014) discovered, the term 'Catalan numbers' has been used systematically in the literature since the publication of Riordan's monograph (Riordan 1968). It is unlikely that it would be now possible to return Euler's name to the Catalan numbers. After all, it would still lead to confusion with the other Euler numbers, i.e. integers occurring in the coefficients of the Taylor series of $1/cosht$.

The Catalan numbers form the sequence A000108 in the OEIS. This sequence occurs in many combinatorial problems and allows many interpretations (Stanley 2015). These numbers found applications even in quantum mechanics (Cohen, Hansen, Itzhaki 2016).

2.5. Mathematics in the correspondence: divergent series

Divergent series belonged to Euler's permanent scientific interests. The results of Euler and Goldbach (including those given in their correspondence) with respect to divergent series are frequently discussed in modern literature (Ferraro 1998). Some authors are critical of Euler's approach:

> Despite being regarded as one of the four greatest mathematicians of all time, to this day Euler's reputation is tarnished because of the views he held on divergent series (Kowalenko 2011, s. 370).

In a letter dated August 7, 1745, Euler mentions his discussion with Nicolaus Bernoulli about divergent series, for the expansion $1 - 1 + 2 - 6 + 24 - 120 + 720 - \ldots$. Bernoulli denied that all such series must have a definite sum, while Euler insisted on the opposite: that every such series must have a definite value. At the same time, Euler insists that this value is not called a sum, since the word 'sum' is associated with 'summation', and summation does not make sense when applied to divergent series. Euler offers the following somewhat loose definition: the sum of each series is the value of the finite expression from which the series is obtained by development.

The methods of series summation have a theoretical justification based on modern science. The set of all numerical series forms a linear space, in which a subset of convergent series is a linear subspace. Since the mapping of each convergent series to its sum is a linear functional on the subspace of the convergent series, this functional can be linearly extended to the space of all series by the Hahn–Banach theorem. The proof of the Hahn–Banach theorem relies on various variants of the axiom of choice and is therefore non-constructive. Thus, the proven existence of methods of summation of divergent series is only theoretical and has no applications (see (Domoradzki 2007)).

Note that a new look at the methods used by Euler for operations with infinite sums and products appeared within the framework of non-standard analysis – a section of mathematics in which the idea of Leibniz and his followers about the existence of infinitely small constant values other than zero and infinitely large constant values is implemented. Operations with infinitely large and infinitely small numbers in the framework of hyperreal numbers make it possible to justify some of Euler's operations with infinite series, which were considered unreasonable (Kanovei, Reeken 1995–96). Among other things, this testifies to Euler's strong mathematical intuition.

2.6. Mathematics in the correspondence: topology

Leonhard Euler is considered the father of topology. Two of Euler's statements are considered the first topological theorems: that on Königsberg bridges and that on convex polyhedra. Euler's well-known formula for convex polyhedra is

$$V - E + F = 2,$$

where V is the number of vertices, E is the number of edges, and F is the number of faces. The formula can be found in Euler's letter to Goldbach of November 14, 1750 (number 149). The formula had already been known for Platonic solids and Euler established it for convex polyhedra. Later, Euler (1758a; 1758b) published a proof of this formula. Note that now there are many proofs of Euler's formula.

This formula is the basis of combinatorial topology. This is a part of topology that studies topological invariants of spaces that can be derived from their combinatorial decompositions, e.g., decompositions of polyhedra into simplicial complexes.

As a generalization of Euler's formula, we obtain the notion of Euler characteristic of a finite simplicial complex

$$\chi = k_0 - k_1 + k_2 - k_3 + \dots,$$

where k_i is the number of i-dimensional simplices in the complex.

The Euler characteristic can also be defined for arbitrary topological spaces by using the Betti numbers b_i, which are the ranks of the i-th singular homology group of the space. By the definition,

$$\chi = b_0 - b_1 + b_2 - b_3 + \dots$$

The Euler characteristic is a homotopy invariant and satisfies the product formula. For compact orientable surfaces, the Euler characteristic can by expressed through the genus (the number of handles in the representation of the surface as a sphere with handles) g by the formula

$$\chi = 2 - 2g.$$

A far generalization of the Euler characteristic is the Euler characteristic classes of vector bundles over topological spaces.

The Gal–Euler's series of a finite simplicial polyhedron is defined as follows:

$$\mathrm{eu}_X(t) = \sum \chi(B(X, n)t^n,$$

where $B(X, n)$ is the space of sets of cardinality n in X (see Farber (2008), where these series are applied to topological robotics).

The Euler characteristic is also used in differential geometry; it appears in the Gauss–Bonnet formula: for any closed two-dimensional Riemannian manifold M

$$\int_M K dA = 2\pi\chi(M),$$

where K is the Gaussian curvature of M and dA is the element of area of M.

It is known that Euler is considered the founder of graph theory. In a letter to Goldbach of April 26, 1757 (number 184), he considers the problem of the move of a knight on a chessboard: it is necessary to pass the board in such a way as to visit each square exactly once. In modern terms, this is the problem of finding a Hamiltonian path in the corresponding graph.

2.7. Mathematics in the correspondence: 'fractal geometry'

Some topics discussed in the correspondence are now classified as belonging to fractal geometry, an area in mathematics which, informally speaking, deals with objects that are locally similar to themselves (Mandelbrot 1982, Hutchinson 1981).

In letters number 55 (from Goldbach to Euler of (September 20) October 1 [*these are Old Style and New Style dates*], 1742) and 56 (from Euler to Goldbach of October 27, 1742), blood corpuscles are mentioned in connection with discoveries made by the Dutch researcher Leeuwenhoek by means of a microscope. He remarked that in the cavity of each such corpuscle there are six smaller balls each similarly containing six smaller balls, etc. Goldbach remarked that the volume of the figure consisting of balls at stage n does not exceed $\frac{6^{n-1}}{\left(1+\sqrt{2}\right)^{3n-3}}$ of the volume of the initial sphere. Therefore, if the process could be iterated infinitely many times, then the volume of the resulting figure should equal 0, which is impossible for physical reasons. In his reply, Euler agrees with this argumentation.

Actually, the limit figure is homeomorphic to the Cantor set and this can be considered as a very first example of a (locally) self-similar set.

2.8. Mathematics in the correspondence: algebra

Part of the correspondence is devoted to algebra. Two main topics can be distinguished: irrationality and roots of polynomials. Algebraic results in Euler's correspondence are briefly discussed in (Więsław 2008). The fundamental theorem of algebra states that every polynomial with complex coefficients has the number of roots (counted with multiplicities) which equals its degree. The real form of this theorem states that every polynomial with real coefficients is the product of linear and quadratic terms. In his letter of December 15, 1742 (number 58), Euler wrote:

> omnem expressionem algebraicam $a + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4$ + etc. vel in factores reales simplices $p + qx$, vel saltem in factores reales quadratos $p + qx + rxx$ resolvi posse.

Euler remarks that, although he cannot prove this statement rigorously, it would be very useful in analysis, for integrating rational functions. Actually, the method consists in resolving denominators of these functions into simple factors; if some of these factors are imaginary, then they can be decomposed into couples so that the product of each couple is a real quadratic trinomial. In connection with this, as Euler remarks, Bernoulli conjectured that the polynomial

$$x^4 - 4x^3 + 2x^2 + 4x + 4$$

is a counterexample to this statement. However, Euler discovered a decomposition of this polynomial thus denying Bernoulli's conjecture:

$$\left(x^2 - \left(2+\sqrt{4+2\sqrt{7}}\right)x + 1 + \sqrt{7} + \sqrt{4+2\sqrt{7}}\right)\times\times$$

$$\times\times\left(x^2 - \left(2-\sqrt{4+2\sqrt{7}}\right)x + 1 + \sqrt{7} - \sqrt{4+2\sqrt{7}}\right).$$

Actually, Euler proved the fundamental theorem of algebra for polynomials of degree < 6. The proof obtained by Gauss in 1799 contained an implicit assumption that every odd-degree polynomial with real coefficients has a real root. Later, Gauss completed his proof but used the methods which are not algebraic (e.g. either analytic or topological). A purely algebraic proof was obtained by P. Błaszczyk (2022); he used an extension of the field of real numbers to the non-Archimedean field of hyperreals via an ultraproduct construction.

One more algebraic topic mentioned in Volume IV is the identity that expressed the product of two sums of four squares as a sum of four squares. Euler discussed it in the letter of May 4, 1748 (number 95). Actually, if $n = a^2 + b^2 + c^2 + d^2$, $m = p^2 + q^2 + r^2 + s^2$, then $mn = A^2 + B^2 + C^2 + D^2$, where $A = ap + bq + cr + ds$, $B = aq - bp - cs + dr$, $C = ar + bs + cp - dq$, $D = as - br + cq - dp$ (Euler's four-square identity). This identity is used to prove Lagrange's four-square theorem: every natural number can be represented as the sum of the squares of four integers. Note also that this identity was later used by Hamilton (1844 to 1850) in his research on quaternions. It expressed the multiplicativity of the norm of quaternion. In connection with this, a historical misunderstanding arose, namely the opinion that it was Euler who discovered quaternions (see (Blaschke 1959) and criticism in (Koetsier 2007)). In fact, Euler's name is connected with the theory of quaternions, the connection is established through the rotation of a body around a fixed point. The rotation of a vector in three dimensions is described by four Euler parameters. Actually, given a unit quaternion $q = a + b\mathrm{i} + c\mathrm{j} + d\mathrm{k}$, the rotation matrix is

$$\begin{bmatrix} 1-2\left(c^2+d^2\right) & 2bc-2ad & 2ac+2bd \\ 2ad+2bc & 1-2\left(b^2+d^2\right) & 2cd-2ab \\ 2bd-2ac & 2ab+2cd & 1-2\left(b^2+c^2\right) \end{bmatrix}.$$

That the composition of the two rotations corresponds to the multiplication of the unit quaternions is a consequence of Euler's four square identity. This identity was known to Olinde Rodrigues who introduced Rodrigues' rotation formula.

The relatively low number of topics in the correspondence devoted to algebra can be explained by the fact that this discipline was just emerging and Goldbach knew much less about it compared, for example, to the number theory. However, as is emphasized in (Więsław 2008), the results obtained are important for the development of algebra since they opened new directions of algebraic investigations.

3. Volume VIII of Series IVA

The editors of the volume were Andreas Kleinert and Thomas Steiner with the participation of Gisela Kleinert and Martin Mattmüller.

From the General Introduction, we learn that the volume contains Euler's correspondence with 14 members of the University of Halle, at that time the largest Prussian university. The entire collection contains 236 letters, of which only 17 were written by Euler. The correspondence took place mainly in the period 1741–1766. Some of the letters to Euler were related to requests for recommendations for obtaining academic positions. As a rule, Euler responded positively to such requests when he was sure of the candidate's sufficient qualifications.

All the correspondence, Introductions, and Editor's comments are in German. Three letters originally written in Latin have been translated into German.

Below we list Euler's correspondents; at the same time, as a rule, we limited ourselves to indicating the total number of letters (and how many of them were written by Euler), as well as the period of correspondence. Correspondents are listed in alphabetical order. Correspondence with each of them is accompanied by a meaningful introduction with biographical information, historical background and an overview of the letters.

- Thomas Abbt (1738–1766): 1 letter to Euler, 1759.
- Beniamin Brauser (1725–?): 4 letters to Euler in the period 1746–1747.
- Johann Peter Eberhard (1727–1779): 1 letter to Euler, 1754.
- Franz Christoph Jetze (1721–1803): 1 letter to Euler, 1752.
- Wenzeslaus Johann Gustav Karsten (1732–1787): 38 letters (15 from Euler), in the period 1758–1765.

In the introduction, the correspondence between Euler and Karsten is analyzed in great detail. Its subjects mainly concerned the following topics: Karsten's professional position, discussions of mathematical problems, Karsten's commitment to print Euler's textbook on mechanics, and his request to Euler to help him purchase books and tools. Karsten felt academically isolated in Rostock, where he lived at the time (there were no other mathematicians there) and asked for Euler's support in getting another position. The mathematical topics discussed by Euler in his correspondence with Karsten included polyhedra, differential equations of three variables, transcendental curves, the motion of solid bodies, and even ballistics.

- Christian Albrecht Körber (1699 – after 1747): 2 letters to Euler, 1741–1742.
- Christian Gottlieb Kratzenstein (1723–1795): 8 letters (1 from Euler), 1747–1752.
  Euler's correspondence with Kratzenstein is reflected in the book (Snorrason 1974). It is likely that Euler helped Kratzenstein to get a position at the St. Petersburg Academy of Sciences.
- Johann Gottlob Krüger (1715–1759): 4 letters to Euler, 1747–1749.
- Johann Joachim Lange (1699–1765): 12 letters to Euler, 1753–1755.
- Johann Adam Osiander (1718–1749): 5 letters (1 from Euler), 1741–1742.
- Johann Ernst Philippi (ca. 1700–1757): 2 letters, 1749–1750.
- Johann Heinrich Schulze (1687–1744): 1 letter to Euler, 1743.
- Johann Andreas von Segner (1704–1777): 154 letters (Euler's letters to Segner were not preserved, as Segner ordered his entire archive to be burned posthumously), 1741–1771.

Segner was engaged in mathematics, mechanics, physics, chemistry and botany. He invented a water turbine called the Segner wheel. Among the most important scientific achievements of Segner, we note the discovery of the main axes of inertia of an absolutely solid body (Euler discovered them three years later).

References to the number of triangulations of a polygon appear in letters 38 and 40 of Segner to Euler.

- Johann Wilhelm von Segner (1738–1795): 2 letters to Euler, 1764.

This volume concludes with a bibliography for the Introductions, letters, and editor's comments.

4. Conclusions

The publication of the volumes of Leonard Euler's correspondence with Christian Goldbach and others is a significant event in the history of mathematics. In the 18th century, when there were no mathematical journals, correspondence was one of the main ways of promoting scientific ideas and results.

This publication is characterized by extremely careful editorial work. This primarily concerns the correspondence with Christian Goldbach, which has been translated into English. Given the current language situation, this will greatly expand the group of researchers, both in mathematics and in the history and philosophy of mathematics, who will be able to work with this material.

At the same time, the correspondence with Goldbach was also published in the original languages, mainly German and Latin, with maximum preservation of the author's style and mathematical notation.

It is clear from the above correspondence that Euler carefully read and commented on the works sent to him by younger scientists. The consequence of this was a broad exchange of mathematical ideas and this had a great impact on the development of mathematics and other sciences at that time.

Bibliography

Blaschke, Wilhelm 1959: Euler und die Kinematik. [In:] Kurt Schröder, S*ammelband der zu Ehren des 250. Geburtstages LEONHARD EULERs der Deutschen Akademie der Wissenschaften zu Berlin vorgelegten Abhandlungen*. Berlin: Akademie-Verlag, pp. 35–41.

Błaszczyk, Piotr 2022: A Purely Algebraic Proof of the Fundamental Theorem of Algebra. *Annales Universitatis Paedagogicae Cracoviensis. Studia Mathematica* 8, pp. 7–23.

Błaszczyk, Piotr; Petiurenko, Anna 2023: Euler's Series for Sine and Cosine: An Interpretation in Nonstandard Analysis. [In:] Zack, M., Waszek, D. (eds.), *Research in History and Philosophy of Mathematics. Annals of the Canadian Society for History and Philosophy of Mathematics/ Société canadienne d'histoire et de philosophie des mathématiques*. Birkhäuser, Cham. DOI: 10.1007/978-3-031-21494-3_5.

Calinger, Roland 1996: Leonhard Euler: The First St. Petersburg Years (1727–1741). *Historia Mathematica* 23, pp. 121–166.

Catalan, Eugène Ch. 1838: Note sur une équation aux différences finies. *Journal des Mathématiques Pures et Appliquées* 3, pp. 508–516.

Catalan, Eugène Ch. 1887: Sur les nombres de Segner. *Rendiconti del Circolo Matematico di Palermo* 1, pp. 190–201.

Cohen, Eliahu; Hansen, Tobias; Itzhaki, Nissan 2016: From entanglement witness to generalized Catalan numbers. *Scientific Reports* 6, 30232. DOI: 10.1038/srep30232.

Denkowski, Maciej P. 2020: Leonhardi Euleri Opera Omnia IVA/7: Commercium Epistolicum (2017, Euler – French Speaking Scientists From Switzerland). Editors: S. Bodenmann, V. Hug, M. Ilić, A. Kleinert. Basel: Birkhäuser, 2017, XII+621 pages – A volume overview. *Studia Historiae Scientiarum* 19, pp. 545–560. DOI: 10.4467/2543702XSHS.20.017.12573.

Domoradzki, Stanisław 2007: On Euler summation method in S. Banach's monograph. [In:] L. Euler and contemporary science, conference materials (In Russian.), Petersbourg, pp. 111–116.

Eneström, Gustaf 1910: Verzeichnis der Schriften Leonhard Eulers. *Jahresbericht der Deutschen Mathematiker-Vereinigung*.

Euler, Leonhard 1758a: Elementa doctrinae solidorum. *Novi Commentarii Academiae Scientiarum Imperialis Petropolitanae* 4 (1752–1753), pp. 109–140, *Opera Omnia* (1) Volume 26, pp. 72–93.

Euler, Leonhard 1578b, Demonstratio nonnullarum insignium proprietatum quibas solida hedris planis inclusa sunt praedita. *Novi Commentarii Academiae Scientiarum Imperialis Petropolitanae* 4 (1752–1753), pp. 140-160 (published 1758) *Opera Omnia* (1) Volume 26, pp. 94–108.

Euleri, Leonhard 2016: *Leonhardi Euleri Opera Omnia* vol. IVA 4: *Commercium Epistolicum cum Christiano Goldbach / Correspondence with Christian Goldbach*. Basel: Springer. Editors: Franz Lemmermeyer, Martin Mattmüller, ISBN: 978-3-0348-0892-7 (Pt. I); 978-3-0348-0880-4 (Pt. II).

Euler, Leonhard 2018: *Leonhardi Euleri Opera Omnia* . Series Quarta A: *Commercium Epistolicum* 8. Cham: Birkhäuser (ISBN 978-3-319-75942-5/hbk). xiv, 713 pp.

Farber, Michael 2008: *Invitation to Topological Robotics*. EMS Press, 143 pp.

Ferraro, Giovanni 1998: Some Aspects of Euler's Theory of Series: Inexplicable Functions and the Euler–Maclaurin Summation Formula. *Historia Mathematica* 25(3), pp. 290–317.

Hamilton, William R. 1844–1850: On quaternions or a new system of imaginaries in algebra. *Philosophical Magazine*. URL: https://www.maths.tcd.ie/pub/HistMath/People/Hamilton/OnQuat/OnQuat.pdf (accessed on: 26 October 2024).

Helfgott, Harald A. 2013: The ternary Goldbach conjecture is true. arXiv:1312.7748 [math.NT] (accessed on: 26 October 2024).

Helfgott, Harald A. 2015: The ternary Goldbach problem. arXiv:1501.05438 [math.NT] (accessed on: 26 October 2024).

Hutchinson, John E. 1981: Fractals and self-similarity. *Indiana University Mathematics Journal* 30, pp. 713–747.

Kanovei, Vladimir; Reeken, Michael 1995–96: Summation of divergent series from the nonstandard point of view. *Real Analysis Exchange* 21(2), pp. 473–497.

Koetsier, Teun 2007: Euler and Kinematics. [In:] Robert E. Bradley, Edward Sandifer (eds.), *Leonhard Euler: Life, Work and Legacy.* Amsterdam: Elsevier B.V., pp. 167–194.

Kowalenko, Victor 2011: Euler and Divergent Series. *European Journal of Pure and Applied Mathematics* 4, pp. 370–423.

Larcombe, Peter J. 1999: The 18th century Chinese discovery of the Catalan numbers. *Mathematical Spectrum* 32(1), pp. 5–7.

Maligranda, Lech 2008: Gustaf Eneström (1852–1923). *Antiquitates Mathematicae* 2, pp. 27–36.

Mandelbrot, Benoît B. 1982: *The Fractal Geometry of Nature*. "Times Books", 468 pp., ISBN: 9780716711865.

Pak, Igor 2014: History of Catalan numbers, arXiv:1408.5711.

Riordan, John 1968: *Combinatorial Identities*. New York: John Wiley.

e Silva, Tomás Oliveira; Herzog, Siegfried; Pardi, Silvio 2014: Empirical Verification of the Even Goldbach Conjecture and Computation of Prime Gaps up to $4 \cdot 10^{18}$. *Mathematics of Computation*, pp. 2033–2060. DOI: 10.1090/S0025-5718-2013-02787-1.

Schnirelmann, Lev G. 1930: On the additive properties of numbers, first published. [In:] *Proceedings of the Don Polytechnic Institute in Novocherkassk* (in Russian), vol. 14, pp. 3–27, and reprinted in *Uspekhi Matematicheskikh Nauk* (in Russian), 1939, no. 6, pp. 9–25.

Snorrason, Egill 1974: *C.G. Kratzenstein, professor physices experimentalis Petropol. et Havn. and his studies on electricity during the eighteenth century*, Odense: Odense University Press. ISBN 87-7492-092-8.

Stanley, Richard P. 2015: *Catalan Numbers*. Cambridge: Cambridge University Press.

Tao, Terence (2014): Every odd number greater than 1 is the sum of at most five primes. *Mathematics of Computation* 83(286), pp. 997–1038. arXiv:1201.6656. DOI: 10.1090/S0025-5718-2013-02733-0, MR 3143702.

Więsław, Witold 2008: Prace Eulera z algebry. *Antiquitates Mathematicae* 2, pp. 121––132.

Yushkevich, Adolph P.; Kopelevich, Judith Ch. 1983: Christian Goldbach (1690––1764) (Russian). Moscow: "Nauka").

Yushkevich, Adolph P.; Kopelevich, Judith Ch. 1994: Christian Goldbach (1690––1764) (German). Basel: Birkhäuser Verlag.