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*TRANSLATIONS*

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# On Symmetry in Physical Phenomena, Symmetry of an Electric Field and of a Magnetic Field

## Abstract

In this work, the classical concept of symmetry limited to geometric objects (figures and solids), which originated from ancient Greece, has been extended to allow for symmetry studies in other types of objects.

By introducing the concepts of *limiting point groups* and *kinematic elements* characteristic for a studied object, it was determined what types of symmetries are exhibited by an electric field and a magnetic field. It was established that in order for a phenomenon to occur, a characteristic symmetry of a medium must be consistent with the characteristic symmetry of the phenomenon occurring in it. It was also determined that the symmetry elements of the causes must be found in the symmetry of their effects.

**Keywords:** *symmetry, dissymmetry, Curie limiting point groups, symmetry of causes and effects, symmetry of physical fields, characteristic symmetry of phenomenon, characteristic symmetry of medium*

(Abstract and keywords by Andrzej Ziółkowski) [(P0)]

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[(P0)] Translator's note: This document contains the English translation of the work of Pierre Curie: Sur la symétrie dans les phénomènes physiques, symétrie d'un champ électrique et d'un champ magnétique. *Journal de Physique Théorique et Appliquée*, 3e série, 1894, 3(1), pp. 393–415. DOI: 10.1051/jphystap:018940030039300.

## O symetrii zjawisk fizycznych, symetrii pola elektrycznego i pola magnetycznego

**Abstrakt**

W pracy klasyczne pojęcie symetrii ograniczone do obiektów geometrycznych (figur, brył), znajdujące swoje źródło w antycznej Grecji, zostało rozszerzone tak, by możliwe było badanie symetrii innych rodzajów obiektów.

Poprzez wprowadzenie pojęcia *granicznych grup punktowych i elementów kinematycznych* charakteryzujących obiekt, którego symetria jest badana, określono, jakiego typu symetrie wykazują pole elektryczne i pole magnetyczne. Ustalono, że aby możliwe było zachodzenie jakiegoś zjawiska, to charakterystyczna symetria ośrodka musi być zgodna z charakterystyczną symetrią występującego w nim zjawiska. Stwierdzono, także, że elementy symetrii przyczyn muszą znaleźć odzwierciedlenie w symetrii wywołanych skutków.

**Słowa kluczowe**: *symetria, dyssymetria, graniczne grupy punktowe, symetria przyczyn i skutków, symetria pól fizycznych, symetria charakterystyczna zjawiska, symetria charakterystyczna ośrodka*

(Abstrakt i słowa kluczowe opracowane przez Andrzeja Ziółkowskiego)

### 1. I think it would be useful to introduce symmetry considerations known to crystallographers into the study of physical phenomena

For example, an isotropic body can be set in a rectilinear or a rotary motion; A fluid can be a medium of vortex motions; A solid can be compressed or twisted; It may be in an electric or magnetic field; Electric current or heat may flow through it; Natural light or light that is rectilinearly, circularly, elliptically, etc. polarized can pass through it. In all these cases, the occurrence of some characteristic dissymmetry is necessary, in every point of the body. Dissymmetries will be even more complex if we assume that several phenomena coexist in the same medium or if these phenomena are caused in a crystalline medium, which already has — due to its structure — a certain dissymmetry.

Physicists often take advantage of the conditions resulting from symmetry, however, they generally pass over defining the symmetry of the phenomenon itself, because quite often the symmetry conditions are simple and almost obvious, a priori[1].

However, in teaching physics it would be better to formulate these problems explicitly, e.g. in electricity research, to find out almost immediately the existence of a characteristic symmetry of the electric field and the magnetic field; we could then use these concepts to simplify many demonstration experiments.

From the point of view of general ideas, the concept of *symmetry* can be compared with the concept of *dimension*: these two fundamental concepts are characteristics for the *medium* in which the phenomenon occurs, and for the *quantity* used to assess the intensity of the phenomenon, respectively.

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The translation has been provided with translator's end notes and a commentary with additional explanations and information aimed at facilitating the correct understanding of the text without the need for a broad query in external resources. The end notes are marked in the translation as (P1), (P2), etc., and their full content is available in the Extended Commentary added below the translated work.

[1] Crystallographers who need to consider more complex cases have developed a general theory of symmetry. In treaties in the field of physical crystallography (which are at the same time actual physical dissertations), the issues of symmetry are exposed with the utmost care. *See* the works of MALLARD (Mallard 1879, 1884), LIEBISCH (Liebisch 1891), SORET (Soret 1893).

Two media with the same dissymmetry are linked by a special connection, from which we can draw physical consequences. The relationship of the same type exists between two quantities of the same dimension. Finally, when certain causes induce certain effects, the elements of symmetry of the causes must find reflection in the symmetry of produced effects. Similarly, in the equation of the physical phenomenon there is a cause-effect relationship between the quantities appearing on both sides of the equation, and these quantities on both sides have the same dimension.

## 2. Recovery operations and symmetry elements

Determining various types of symmetry can be divided into two large areas, depending on whether it is about determining the symmetry of a limited system or a system that can be considered unbounded. We will only deal with a limited system here[2].

Consider a system defined using analytical data and three orthogonal coordinate axes, for example. The system will have some *symmetry* (P1) when upon using other orthogonal axes of coordinates it will still be defined by the same analytical data.

Elements (points, lines, planes, etc.) defined by means of the same analytical data referred to such different triads of coordinate axes are *homological elements* or *the elements of the same type*.

The operation which makes the transition from the first system to the second is a *recovery operation*[3] (P2).

There are two types of orthogonal triads of coordinate axes symmetrical relative to each other. We will have a *recovery operation* of the system *of the first type* when such an operation is a transition from one triad of axes to the other identical triad of axes. The operation is therefore equivalent to simple displacement in space (P3). The repetition of the same elements in the system takes place.

We will have a *recovery operation of the second type or symmetric transformation in the right sense*, when the operation is the transition from one triad of axes to another one symmetric to the first.

The system is then identical to its image obtained by mirror reflection.

It can be easily demonstrated that during the recovery operation of a limited system at least one point always stays constant in space. It follows that the establishment of all possible types of symmetry of the limited system comes down to establishing *all types of symmetry around the point* which is the center of the shape of the system.

The recovery operations of the first type can always be obtained by a simple rotation around the *repetition axis* (more generally called the *symmetry axis*) passing through the point. The axis of degree $q$ (where $q$ is an integer number) will give recovery (translator's note:

[2] The theory of structure of crystalline bodies is nothing else but the general theory of *symmetry of an unlimited medium* with a periodic structure. This is an admirable theory that was developed by BRAVAIS (Bravais 1866), JORDAN (Jordan 1868a; 1868b) and FEDOROW (Fedorow 1891, 1892). Recently, SCHŒNFLIES published a great didactic treaty dedicated to this theory, *Krystallsysteme und Krystallstruktur* (Schœnflies 1891).

Crystalline bodies can be divided into 32 classes (translator's note: *point crystallographic groups* or equivalently *crystallographic classes*), if we consider only symmetries of the external shape; but the theory predicts 230 different types of symmetry for the internal structure of these substances (translator's note: *spatial crystallographic groups*). If all these types exist in nature, it is a real wealth for physicists, because they have 230 media with various symmetries at their disposal.

[3] Shuffling transformation according to German crystallographers.

overlapping, invariance, stability) of the system at rotation angles $0, 1, 2, \ldots, (q-1)$ times $2\pi / q$ (translator's note: $q$-fold axis, *e.g. two-fold, three-fold etc.*).

We will consider the *direction and sense* of each axis of the system, which doubles the number of axes, because for one axis we will count two directions with opposite sense. If these two axes with opposite sense are of different type from the point of view of repetitions (for example, the axis of the regular pyramid) and of the degree $q$, we will mark them by $(L_q\ l_q)$ (P4).

If these two axes with opposite sense are of the same type with respect to repetitions (for example, the main axis of the prism) and of the degree $q$, we will mark them by $(2L_q)$. We then have a *double axis* (P5). In this case, in the system, by necessity, there exists an axis of repeatability with an even degree perpendicular to the double axis, which allows its transformation into itself by rotation by 180°, which is the element of the recovery operation of the system.

The recovery operations of the second type can always be obtained by mirror reflection, which is accompanied by rotation around the axis normal to the plane of mirror reflection.

Several cases should be examined:

1° Rotation is zero; we have a simple mirror reflection and the system has a *plane of symmetry* (P).

2° Rotation is 180°; we have a *center of symmetry* (C).

3° Axis normal to the reflection plane is the axis of repetitions of degree $q$ and we have $q$ symmetric transformations; each of these operations consists of one mirror image, which is followed by one of the rotations

$$0, \quad \frac{2\pi}{q}, \quad 2\frac{2\pi}{q}, \quad \ldots \quad , \quad (q-1)\frac{2\pi}{q};$$

We then have a *simple plane of symmetry of the degree* $q$, which we will mark by $P_q$.

4° Axis normal to the reflection plane is the axis of repetitions of degree $q$ and we have $q$ symmetric transformations; each of these operations consists of a mirror image, which is followed by one of the rotations

$$\frac{1}{2}\frac{2\pi}{q}, \quad (1+\frac{1}{2})\frac{2\pi}{q}, \quad (2+\frac{1}{2})\frac{2\pi}{q}, \ldots, (q-1+\frac{1}{2})\frac{2\pi}{q}$$

around the axis. We then have an *alternative plane of symmetry of degree* $q$; we will denote it by $\pi_q$.

The model shown in Figure 1. has an axis of degree 4 with a plane $P_4$ of a simple symmetry of degree 4. Four lower arrows are obtained by a simple mirror reflection of the four upper arrows and vice versa. The system can be recovered by a simple mirror reflection and the accompanying rotation by 90° repeated a certain number of times.

The model in Figure 2. has an axis of the 4th degree and an alternative symmetry plane $\pi_4$ of the 4th degree, perpendicular to the direction of the axis. Four lower arrows differ in location with images of the four upper arrows obtained by a simple mirror reflection. The system can be recovered by mirror reflection, followed by rotation by 45° degrees an odd number of times.

It is worth noting that the model in Figure 2. is overlayable on its mirror image, although it has neither a plane nor a center of symmetry. There is only an alternative symmetry plane[4].

### 3. Groups of recovery operations

All recovery operations of the system are defined by the use of the symmetry elements that we have just listed.

A *group* of recovery operations will be a combination of operations in such a way that any two operations carried out successively will give the same result as the one which is obtained through a single operation included in the group.

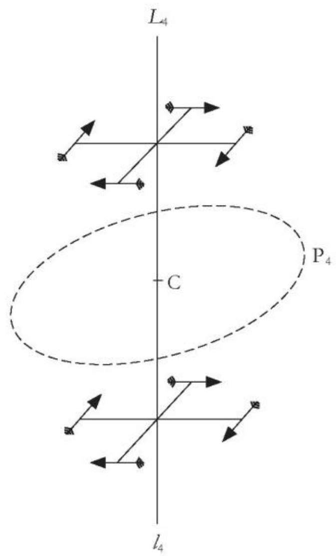


Fig. 1.

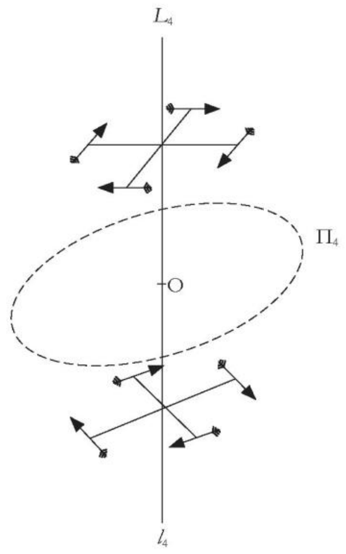


Fig. 2.

Here we give the full table of all groups of recovery operations relative to the point. These operations are completely defined by listing the elements of symmetry.

We can see that groups of symmetry elements can be divided into *seven classes* which differ from each other in the character of the group of axes which they contain (translator's note: The division into symmetry classes introduced below is based on different criteria compared to the criteria of classic division into crystallographic classes, although it is analogous to it.). Each class can exist with or without symmetric transformation in the proper sense (translator's note: i.e., in the understanding of P. Curie, the mirror image). Usually there are several ways to give symmetry in the proper sense to a group that contains nothing but axes. We obtain a total of 19 families $f$. Let us consider, for example, the class III and assume that $q=3$, we will have a group of axes $2L_3, (3L_2, 3L'_2)$, i.e. a double main axis of degree 3, and three 2-fold axes and those with the opposite sense of the different type $3(L_2, L'_2)$; these three axes are perpendicular to the main axis and they form angles of 120° among them. This system can exist without any other element of symmetry [family (8), the crystalline form of quartz],

[4] P. CURIE (Curie 1884, pp. 89, 418).

Table 1 [P6]

| Class | Axes of repeatability (rotational symmetry) | Family | Symmetry transformations | N | Examples (crystallographic system) | International notation |
|---|---|---|---|---|---|---|
| I. | (No symmetry axis) | 1 | O | 1 | | 1 |
| | | 2 | P | 2 | | $m$ |
| | | 3 | C | 2 | parallelepiped (triclinic) | $\bar{1}$ |
| II. | $L_q l_q$<br>(Axis and its inversion) | 4 | O | $q \quad q=2$ | wine acid (monoclinic) | 2 |
| | | 5 | $P_q$ | $2q\begin{cases} q=2 \\ q=6 \\ q=\infty \end{cases}$ | gypsum (monoclinic)<br>apatite (hexagonal)<br>magnetic field | $2/m$<br>$6/m$<br>$\infty/m$ |
| | | 6 | $\pi_q$ | $2q\begin{cases} q=\infty \\ q=3 \end{cases}$ | magnetic field<br>dioptase (trigonal) | $\infty/m$<br>$\bar{3}$ |
| | | 7 | qP | $2q\begin{cases} q=3 \\ q=\infty \end{cases}$ | tourmaline (trigonal)<br>electric field, cut cone | $3m$<br>$\infty m$ |
| III. | $2L_q$, $qL_2$, $ql'_2$<br>(Double main axis) | 8 | O | $2q\begin{cases} q=3 \\ q=\infty \end{cases}$ | quartz (trigonal)<br>twisted fiber | 32<br>$\infty 2$ |
| | | 9 | $P_q$, qP | $4q\begin{cases} q=3 \\ q=\infty \end{cases}$ | prisms with a triangular base<br>cylinder | $\bar{6}2m$<br>$\infty mm$ |
| | | 10 | $\pi_q$, qP | $4q\begin{cases} q=\infty \\ q=3 \\ q=2 \end{cases}$ | cylinder<br>rhombohedron, spar (trigonal)<br>scalenohedron (tetragonal) | $\infty mm$<br>$\bar{3}m$<br>$\bar{4}2m$ |
| IV. | $4(L_3 l_3)$, $6L_2$<br>(axes of regular tetrahedron) | 11 | O | 12 | sodium chlorate (cubic) | 23 |
| | | 12 | $4\pi_3$, $3P_2$, C | 24 | pyrite (cubic) | $m\bar{3}$ |
| | | 13 | $3\pi_2$, 6P | 24 | regular tetrahedron,<br>zinc blende ZnS (cubic) | $\bar{4}3m$<br>$\bar{4}3m$ |

| | | | | | | |
|---|---|---|---|---|---|---|
| V. | $6L_4, 8L_3, 12L_2$ | 14 | O | 24 | cuprite (cubic) | $4/m\bar{3}2/m$ |
| | (axes of cube) | 15 | $3P_4, 4\pi_3, 6P_2$, C | 48 | cube, octahedron | $4/m\bar{3}2/m$ |
| VI. | $12L_5, 20L_3, 30L_2$ | 16 | O | 60 | | |
| | (axes of regular icosahedron) | 17 | $6\pi_5, 10\pi_3, 15P_2$, C | 120 | regular icosahedron,<br>regular dodecahedron | $\bar{5}\bar{3}m$<br>532 |
| VII. | $\infty L_\infty$ | 18 | O | $\infty$ | sphere filled with fluid with vortices | $\infty\infty$ |
| | (axes of sphere) | 19 | $\infty P_\infty$, C | $\infty$ | sphere | $\infty\infty m$ |

In Table 1.:

$(L_q, l_q)$ denotes axis of degree (multiplicity) $q$, and axis with opposite sense of a different type,

$(2L_q)$ denotes double axis of degree $q$, $C$ is center of symmetry, $P$ is plane of symmetry,

$P_q$ is a simple plane of symmetry of degree $q$,

$\pi_q$ is alternative plane of symmetry of degree $q$.

(Translator's note: $O$ denotes 'auxiliary point'; this is not a center of symmetry, see Figure 2. Information about the crystallographic system in the last but one and the last columns with the naming according to the Hermann-Mauguin (H-M) classification were added by the translator.)

or with a plane of symmetry of degree 3 perpendicular to the main axis $(P_3)$ and 3 symmetry planes $3P$ containing the main axis and 2-fold axes [family (9) prism with a triangular base]. We can still have a symmetric system [family (10), rhombohedron] with an alternative plane of symmetry $\pi_3$ perpendicular to the main axis, 3 symmetry planes containing the main axis and perpendicular to the 2-fold axes and with the center of symmetry.

Each family of classes II and III contains an infinite number of groups, $q$ can be any integer. Families of other classes contain only one group.

In families (5) and (9), there is a center of symmetry when $q$ is even. In families (6) and (10), there is a center of symmetry when $q$ is odd.

In class III, axes $L_2$ and $L'_2$ coincide, but have opposite senses if $q$ is odd. On the contrary, we have 2-fold, double axes of two different types if $q$ is even.

The numbers $N$ determine the class of each group. The $N$ specifies the number of homologous points (translator's note: equivalent configurations) between them in a system, when the points considered are not located on any axis or on any plane of symmetry. The $N$ is also the number of orthogonal triads of coordinate axes in which the system looks the same.

Systems with symmetry of families 1, 4, 8, 11, 14, 16, 18, which contain only axes, cannot be overlaid onto their image obtained by mirror reflection; they have *enantiomorphic dissymmetry*[5]. (P7)

A very important concept, from the point of view of our current interests, is the concept of *subgroups* (translator's note: The word *intergroupe* appearing in the original was replaced by *subgroup* as used nowadays). A group of symmetry elements is a subgroup of a wider group of symmetry, when all recovery operations from the first group are part of recovery operations of the second.

For example, family (13) with tetragonal symmetry is a subgroup of family (15) of cubic symmetry. The group $(L_6, l_6)$, $6P$ of family (7) (symmetry of a regular hexagonal pyramid) is a subgroup of group $\frac{2L_6}{P_6}, \frac{6L_2, 6L'_2}{6P_2}, C$ of family (9) (regular hexagonal prism). Family (4) is a subgroup of families (5), (6), (7), (8), (9), (10) for the same value $q$, etc.

## 4. Characteristic dissymmetry of physical phenomena

Let us now consider any point of the medium in any physical state. Symmetry at this point will necessarily be characterized by one of the groups from Table 1.[6].

We will formulate the following theorems:

[5] Detailed information can be found in treaties on crystallography. See also BRAVAIS (Bravais 1866), JORDAN (Jordan 1868), P. CURIE (Curie 1884, p. 418).

[6] Some minds may hesitate before applying to the medium in any physical state the classification which was for the first time determined from the point of view of pure geometry. We will note that we can bring all the reasoning, which is used for reestablishing groups, to the following form: let A, B, C be three triads of the orthogonal axes of coordinates, in which the system presents itself the same, let D be the fourth system of orthogonal coordinate axes, which is placed relative to C, just the same as B relative to A; D will continue to be a triad of the coordinate axes in which the system will present itself as in A, B, C. The way of reasoning does not prejudge anything about the nature of the system.

> *The characteristic symmetry of a phenomenon is the maximum symmetry consistent with the occurrence of that phenomenon.*
>
> *A phenomenon can occur in a medium that has the characteristic symmetry of the phenomenon or the symmetry of one of the subgroups of its characteristic symmetry.*

In other words, certain elements of the symmetry of the medium may, but do not have to, co-occur with certain symmetries of the phenomenon. What is necessary is that some elements of symmetry do not occur. *It is dissymmetry what generates the occurrence of a phenomenon*.

It would be much more logical to name a plane of dissymmetry any plane that is not a plane of symmetry; the axis of dissymmetry each axis which is not the axis of symmetry, etc., and in general to provide a list of operations that are not recovery operations in a given system. It is these operations that indicate the existence of dissymmetry and, as a consequence, the possibility of occurrence of some feature in the system. But in the groups considered, there is an infinite number of operations that do not lead to recovery of the system and in general a finite number of recovery operations; that is why it is much easier to provide a list of these last operations.

We also see that when several phenomena of a different nature superimpose on each other creating one system, then the dissymmetries add up. Then, in the system only those elements of symmetry remain that are common to all phenomena considered separately (translator's note: The above statement is known as the ***Principle of Superposition of Dissymmetries***.

> *Since certain causes produce certain effects, the elements of symmetry of the causes must find reflection in the elements of symmetry of the caused effects.*
>
> *When certain effects exhibit a certain dissymmetry, this dissymmetry must manifest itself in the causes that generated these effects.*

The opposite statements to the ones formulated above are not true, at least in practice, that is the produced effects can be more symmetric than the causes that induce them.

Some dissymmetries of causes may not affect certain phenomena or at least have an impact too weak to take them into account, which boils down in practice to the same as if such an impact did not exist.

It is interesting, from the point of view of physical phenomena, to consider separately the *groups having an axis of isotropy*. There are five such groups; we will denote them by (a), (b), (c), (d) and (e) (translator's note: See also Figure S4 and Figure S5 in the translator's Commentary).

$(a)$ $\frac{2L_\infty}{P_\infty}, \frac{\infty L_2}{\infty P_2}, C$ — E.g.: *Cylinder, Body compressed in one direction*

$(b)$ $2L_\infty, \infty L_2$ — *Twisted Cylinder*

$(c)$ $(L_\infty l_\infty), \infty P_1$ — *Cut cone, Electric field*

$(d)$ $\frac{L_\infty l_\infty}{P_\infty}, C$ — *Rotating cylinder, Magnetic field*

$(e)$ $(L_\infty l_\infty)$ — *Rotating cut cone*

***Cylindrical group*** **(a)**, the most symmetric, has elements of the circular cylinder symmetry, i.e. a double axis of isotropy $2L_\infty$, with an infinite number of 2-fold, double axes

$\infty L_2$ perpendicular to the main axis and passing through the center of shape of the system, a plane of simple symmetry $P_\infty$ of degree $\infty$ orthogonal to the main axis, an infinite number of planes of simple symmetry $\infty P_2$, of degree $2$, containing the main axis, and the center of symmetry $C$.

If an isotropic body is squeezed in a certain direction, it becomes anisotropic and has symmetry of the cylindrical group (a). It is known that the body squeezed in this way has optical properties such as crystals with the optical axis; symmetry (a) is exactly the maximum symmetry compatible with the occurrence of this phenomenon. Crystalline bodies with the optical axis have symmetries which are subgroups of symmetry (a).

The remaining groups (b), (c), (d) and (e), with the axis of isotropy, are subgroups of the cylindrical group (a).

***Group* (b)** always has a double isotropy axis and 2-fold axes, but it no longer has a center or planes of symmetry. Group (b) is a holoaxial subgroup of group (a). Group (b) has symmetry of cylinder or fiber, twisted around its axis; (translator's note: See Figure S4 in the translator's Commentary). It is symmetry of the center of shape of a system created from two identical cylinders with axes in one line rotating around their axes with equal angular velocities in opposite directions. *Torsional symmetry* (b) does not have other symmetry elements except the axis of repeatability (axis of rotational symmetry); it has non overlayable dissymmetry (enantiomorphism), which is necessary for the occurrence of the phenomenon of ordinary rotary polarization of active bodies. It can also be said that symmetry (b) can be obtained if the cylinder is filled with a liquid having the property of rotary polarization. The crystalline form of quartz $2L_3$, $3(L_3 L_2')$ has symmetry of subgroup of group (b).

***Group* (c)** has an axis of isotropy and the one with the opposite sense of a different type $(L_\infty\, l_\infty)$; this axis is therefore no longer double (in other words, this axis is no longer equivalent regarding rotation operation). Group (c) still has an infinite number of symmetry planes containing an isotropy axis, but it already has neither plane of symmetry orthogonal to the axis, nor the center of symmetry, nor the 2-fold axes of the cylindrical group. It is symmetry of any point on the axis of the circular cut cone (translator's note: See also Figure S4 in the translator's Commentary). It is symmetry of force, velocity, field of universal gravitation; it is finally *symmetry of electric field*. All these phenomena are represented, very aptly from the point of view of symmetry, by an arrow.

Let us consider, for example, the field of universal gravity. The material sphere *M* with the center at point *O* acts on the external point *A* by generating there a field of Newton's attraction. If we assume that the material from which sphere *M* is made by itself does not introduce any dissymmetry, we can see that axis *OA* is the axis of isotropy such that every plane passing through *OA* is a plane of symmetry, and these are the only elements of symmetry passing through point *A*. This is symmetry of group (c). Hence, it follows that the Newtonian attraction field may occur in a medium with symmetry (c) or one of its subgroups; what is more, one cannot imagine that the symmetry of the medium could be greater than (c), because in such a case it would have to be the symmetry of the cylindrical group (a) or the symmetry of the sphere (19) see Table 1., and the field could not have a *sense*, and it would be the same with forces and velocities. If we put the material sphere at point *A*, then force will act on the matter. The body will then be able to go into a state of motion in direction *AO*, reach a certain velocity, and nothing in this process will disturb the symmetry of the system. Therefore, symmetry (c) at the same time represents the symmetry of force acting on ponderable matter and the symmetry of ponderable matter accelerated to a specific velocity.

In order to determine the symmetry of the electric field, let us assume that this field is produced by two round plates made of zinc and copper facing opposite to each other, similarly as plates of the air capacitor. Consider a point between two plates lying on a common axis; we see that this axis is an axis of isotropy and that every plane containing this axis is a plane of symmetry. Elements of symmetry of causes should be found in the produced effects; therefore, the electric field is compatible with symmetry (c) and its subgroups.

Group (a) of cylindrical symmetry and family (19) of spherical symmetry are the only groups containing subgroup (c). It is therefore unlikely that the electric field has a greater symmetry than (c). This last point can be shown rigorously if we assume that the force acting on the ponderable body has a group (c) as a characteristic symmetry, as we saw above. Let us assume that there is an insulated, conductive sphere charged with electricity, and then for some reason electric field appears. A force will start to act on the ball in the direction of this field. Dissymmetry of this action should be sought in the causes that induced it; since the force does not have an axis of symmetry perpendicular to the direction of its action, the system of charged sphere and field also cannot have this element of symmetry. However, the charged sphere considered separately from the field has isotropy axes in all directions; Thus, the dissymmetry in question can only be caused by an electric field which cannot have an axis of symmetry perpendicular to its direction. Therefore, the electric field cannot get cylindrical or spherical symmetry, and its characteristic symmetry is the symmetry of group (c). The symmetry of electric current and dielectric polarization is necessarily the same as the symmetry of the field that causes these phenomena.

Piroelectric and piezoelectric phenomena are a new confirmation of previous conclusions on the characteristic symmetry of the electric field. The crystal of tourmaline, for example, polarizes electrically in the direction of its 3-fold axis when heated or squeezed in the direction of this axis. Gold, when heated or squeezed, in no way changes its crystalline symmetry, which is $(L_3, l_3)3P$, i.e. a 3-fold axis (and axis with the opposite sense of a different type), which is contained by three planes of symmetry; it is symmetry of subgroup (c) $(L_\infty, l_\infty)\infty P$, so this symmetry is compatible with the occurrence of *dielectric polarization along the axis*.

Finally, let us notice that the electric field causes the same optical phenomena in liquids that are obtained by squeezing in solids (Kerr phenomenon). The characteristic symmetry of these phenomena is cylindrical symmetry (a), of which group (c) is a subgroup; therefore, we see that the phenomenon of Kerr reveals only a part of the characteristic dissymmetry of the electric field. The phenomenon of electrical dilatation (Duter phenomenon) reveals only the dissymmetry of group (a).

***Group*** **(d)** has an isotropy axis and an axis with opposite sense of the other type $(L_\infty\ l_\infty)$; thus, this axis is not a double axis with respect to repetition operation (translator’s note: here: rotation), but the system has a center of symmetry and a plane of symmetry of degree $\infty$ perpendicular to the axis. Therefore, axes $L_\infty$ and $l_\infty$ with opposite senses are symmetric relative to each other, and it can be said that the axis of isotropy is double by symmetry. The group has neither 2-fold axes nor symmetry planes containing the main axis of the cylindrical group (a). Group (d) determines the symmetry of the center of the shape of circular cylinder, which rotates around its axis at some speed. Again, we must refer to this symmetry in the case of torque, angular velocity and *magnetic field*.

Let us determine, for example, the characteristic symmetry of the magnetic field. For this purpose, consider the magnetic field which exists at the center of the circumferential circuit through which electric current flows; this field is directed perpendicular to the circumference plane. Let us determine the symmetry of the cause, i.e. the symmetry of the center of the circuit through which the current flows. First of all, we have an axis of isotropy perpendicular to the plane of the current flow circuit. Electric current is compatible with the existence of symmetry planes containing the direction of current flow; therefore, the circumference plane will be a plane of symmetry; electric current does not allow the existence of either a repetition axis or a plane of symmetry perpendicular to its direction. Therefore, there is no axis of symmetry in the plane of the circuit or planes of symmetry containing the axis of isotropy. Thus, the symmetry of the causes is a group of symmetry (d) $(L_\infty \, l_\infty)/P_\infty, C$ . These elements of symmetry are compatible with the existence of a magnetic field passing through the axis of isotropy, because the elements of symmetry of the causes are in the produced effects.

We see that the magnetic field can have a plane of symmetry perpendicular to its direction. In addition, the magnetic field does not allow the existence of 2-fold axes of symmetry perpendicular to its direction. To prove this, we will use the phenomenon of induction. Let us consider, for example, a straight wire moving at a certain velocity perpendicular to its direction. Such a system has a 2-fold axis in the direction of velocity. Let us assume that there is a magnetic field in a direction perpendicular to the direction of the wire and the velocity of motion; an electromotive force will appear in the wire. This phenomenon is incompatible with the existence of a 2-fold axis oriented in the direction of the motion, i.e. perpendicular to the wire. Dissymmetry of effects should be found in the causes; the necessary disappearance of the 2-fold axis of symmetry we were talking about may only come from the presence of a magnetic field; the latter cannot therefore have a 2-fold axis of symmetry perpendicular to its direction. (The same argument can be carried out by considering a circular circuit perpendicular to the magnetic field. It could be assumed that this circuit expands without changing its shape, causing an induction current.)

Cylindrical groups (a) and spherical groups (19) have, as a subgroup, symmetry (d), but the existence in these groups of axes perpendicular between them shows that they are not appropriate to describe the symmetry of the magnetic field. The magnetic field is therefore compatible only with group (d) and its subgroups[7].

The phenomenon of magnetic rotational polarization additionally confirms this conclusion[8].

The magnetically polarized body has the same symmetry as the magnetic field.

The phenomena of magnetic dilatation of iron reveal only the dissymmetry of the cylindrical group (a), of which (d) is a subgroup.

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[7] P. Curie (Curie 1884, p. 418, 1893). Lord Kelvin conjectured that magnetization was caused by a defomation of a special medium. This deformation is simply a rotation, which in this very special medium causes the appearance of a counteracting elastic moment. See: Translation of *Lectures of Sir Thomson*, Note of M. BRILLOUIN. This concept is completely consistent with the above symmetry.

[8] To properly deal with the problem of rotary polarization from the point of view of symmetry, it is necessary to introduce elements of symmetry characteristic for unlimited media that we did not talk about. For example, the body through which passes the circularly polarized light has a spiral axis of isotropy.

A large number of crystals are characterized by groups of symmetry, which are subgroups of magnetic symmetry, e.g. apatite $(L_6\, l_6)/P_6, C$, gypsum, iron chloride, or amphibole $(L_2\, l_2)/P_2, C$. It is possible that these crystals were naturally magnetized as a result of their structure. I tried unsuccessfully to determine this polarization through experiments.

Usually, the magnetic field is presented with an arrow; such a representation, often not leading to misunderstandings, is incorrect from a specific point of view of symmetry, because the magnetic field does not change as a result of mirror reflection relative to the plane perpendicular to its direction and changes its sense on the mirror reflection relative to the plane containing its direction. In the case of an arrow representation, it is exactly the opposite.

***Group* (e)** has only an axis of isotropy $(L_\infty\, l_\infty)$, not a double one. Group (e) is a subgroup common to four groups of symmetry (a), (b), (c) and (d); it has conjoined dissymmetries of all these four groups. Therefore, it is consistent with the existence of phenomena whose characteristic symmetry is any characteristic symmetry of the remaining four groups. The group has enantiomorphic dissymmetry.

Five groups (a), (b), (c), (d) and (e) are related to each other like symmetry types of the same crystallographic system. If we borrow the language of crystallographers, then we will say that group (a) gives a full or holohedral symmetry of the cylindrical system. Group (b) corresponds to holoaxial hemihedry (slanted hemihedry or enantiomorphic hemihedry). Group (c) is hemimorphic hemihedry (hemihedry with unparallel walls). Group (d) is parahemihedry (hemihedry with parallel walls); Finally, group (e) corresponds to tetartohedry. [P8]

Although each group contains an infinite number of recovery transformations, yet we can say that groups (b), (c) and (d) contain only half, and group (e) only a quarter of the recovery transformations of group (a).

The models shown in Figures 3, 4, 5, 6 and 7 use the orientation of the arrows to define subgroups, with the main axis of the 4th degree, of groups (a), (b), (c), (d) and (e).

*Figure* 3 [family (9) $q = 4$] shows the subgroup of the cylindrical group (a); it is symmetry of a simple prism with a square base. Four symmetry planes pass through the main axis, two of the first type pass through arrows, the other two of the seconds type are bisectors of angles formed between the previous two. The locations of double 2-fold axes $L_2$ and $L_2'$ and plane $P_4$ perpendicular to the axis are shown in the Figure.

*Figure* 4 [family (8) enantiomorphic, $q = 4$] shows a subgroup of torsional symmetry (b); it is symmetry of strychnine sulfate crystal.

*Figure* 5 [family (7), $q = 4$] shows a subgroup of symmetry of the electric field (c); there are four planes of symmetry passing through the axis: it is exactly the type of symmetry of the calamine crystal which is both piezoelectric and pyroelectric.

*Figure* 6 [family (5), $q = 4$] shows a subgroup of magnetic symmetry (d); it is symmetry of scheelite and erythrite crystals (translator's note: $CO_3(AsO_4)_2\ 8H_2O$).

Finally, *Figure* 7 [family (4) enantiomorphic, $q = 4$] shows a subgroup of group (e), with an isotropic axis (penta-erythrite crystal). In Figure 7, the arrows at the bottom relate to the magnetic field dissymmetry, arrows at the top relate to the electric field dissymmetry. The combination of these arrows leads to the idea of torsional dissymetry, because the rotational motion around the axis in the direction of the bottom arrows – at the same time moving parallel to the axis in the direction of the top arrows – would describe the spiral.

$\frac{2L_4}{P_4}, \frac{4L_2}{2P_2}, \frac{4L_2'}{2P_2'}, C,$

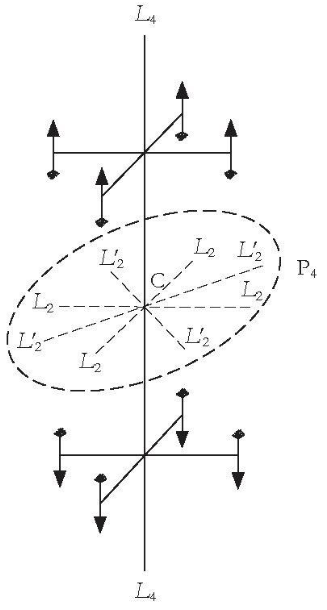


Fig. 3.

$2L_4, 4L_2, 4L_2',$

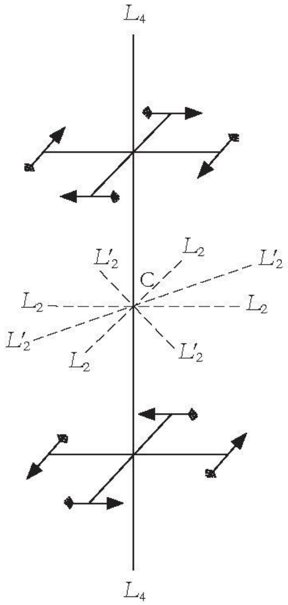


Fig. 4.

$L_4 l_4, 2P, 2P',$

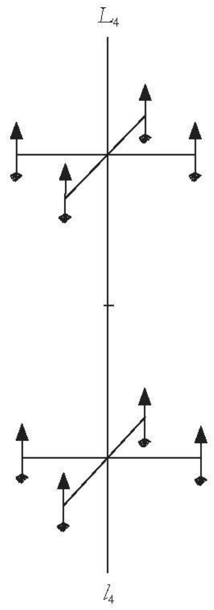


Fig. 5.

$\frac{L_4 l_4}{P_4}, C,$

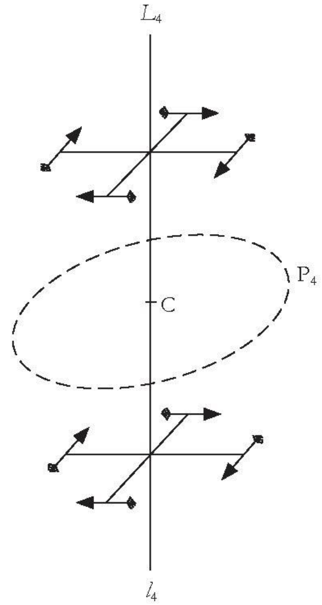


Fig. 6.

$L_4 l_4$

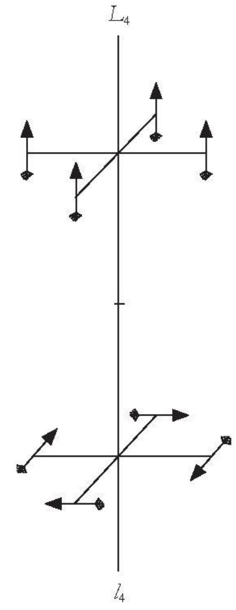


Fig. 7.

### 5. Superposition of causes of dissymmetry in the same medium

When two phenomena of a different nature coexist in the same medium, their dissymmetries add up. If we superimpose the causes of dissymmetry of two of the three groups (b), (c), (d), in such a way that the isotropy axes coincide, we will receive a group (e), because the axis of isotropy will be the only element of symmetry common to the two superimposed groups. Or, to put it differently, group (e) has conjoined dissymmetry of these three groups. So, putting together, as we start calling it, the causes of the dissymmetry of two out of the three groups (b), (c), (d), we will obtain the characteristic dissymmetry of the third group.

Suppose, for example, that we simultaneously apply to the body the *electric field* (c) and the *magnetic field* (d) *with the same direction*, then only the axis of isotropy will remain; the presence of an electric field excludes the existence of a center and a plane of symmetry perpendicular to the axis, and the presence of a magnetic field enforces the disappearance of symmetry planes containing the axis. Thus, the symmetry of the system is symmetry (e) which is a subgroup of symmetry (b): we will have torsional dissymmetry in the body. If we take, for example, an iron wire and magnetize it along its length, then an electric current passing through it will cause the wire to twist (Wiedemann's experiment).

Perhaps it is possible to create a medium capable of exhibiting torsional polarization of active bodies by applying an electric field and a magnetic field in the symmetric body. At least this would not be contradictory to the conditions of symmetry. In the direction of the axis, a superposition could occur of the phenomenon of magnetic rotational polarization (i.e. change of sense along with the change in the direction of light propagation) and the phenomenon of ordinary rotational polarization. Perpendicular to the axis, there could be obtained a pure phenomenon of ordinary rotational polarization. Such a medium with enantiomorphic symmetry would perhaps still enable the realization of certain dissymmetric chemical reactions or the separation of the right and left substances in a racemic mixture or even depositing from a solution of substances in a unique form with symmetric molecules, such as sodium chlorate which usually deposits in the form of dissymmetric mixed right and left crystals.

On the contrary, the electric field or the magnetic field acting individually may not cause a dissymmetric reaction, because these phenomena are consistent with the existence of a plane of symmetry.

Let us assume that we will superimpose a *torsional dissymmetry* (b) and *magnetic dissymmetry* (d), we will again receive symmetry (e) which is a subgroup of the symmetry of the electric field (c).

Let us take a piece of wire, magnetize it and twist it. When the twist occurs in the wire through which the electric current flows, an electromotive force appears if it is arranged in a closed circuit (Wiedemann's experiment).

Symmetry conditions show us that it may happen that bodies with dissymmetric molecules (capable of ordinary rotation) will be dielectrically polarized when placed in a magnetic field.

Finally, let us assume that we apply *torsional dissymmetry* (b) *and electric field* (c); we will again have symmetry (e) which is a subgroup of magnetic symmetry. The iron wire, through which the electric current flows, is magnetized in the direction of its length when it is twisted (Wiedemann's experiment).

The conditions of symmetry allow us to imagine that the body with dissymmetric molecules maybe will undergo magnetic polarization after placing it in the electric field.

### Hall Effect

Let us apply an electric field (c) and a magnetic field (d) in the same medium, with the direction of both fields at a right angle to each other. In this situation, the only element of symmetry common to both fields is the symmetry plane containing the direction of the electric field and perpendicular to the direction of the magnetic field. Therefore, for the entire symmetry we will have plane $P$ [group 4].

On both sides of the plane, the phenomena will have to be symmetric, but in the plane, the symmetry no longer indicates the existence of any constraints. Consider, for example, three orthogonal axes and a rectangular metal plate perpendicular to axis $Ox$ which passes through the center of its shape, and whose sides are parallel to the other axes $Oy$ and $Oz$. If the current flows through the plate along axis $z$, then it cannot be an electromotive force along axis $y$, because plane $zOx$ is a plane of symmetry for the electric current and the plate. If there is no electric current, but along axis $x$ perpendicular to the plate there is a magnetic field, then there can be no current along axis $y$, because axis $x$ is 2-fold axis for the field and the plate, and furthermore there is a center of symmetry. If we now have both a magnetic field along axis $x$ and electric current along axis $z$, then the axis, the center, and the plane of symmetry disappear and nothing obstructs anymore, from the point of view of symmetry, the electromotive force to appear along axis $y$.

The theory of heat propagation and electricity in crystalline bodies (Stokes, Thomson, Minnigerode, Boussinesq) shows that for certain crystalline media, the so-called *rotation* coefficients must be taken into account. This applies to crystals from family (5) $(L_q\ l_q)/P_q$ and (6) $(L_q\ l_q)/\pi_q$ and their subgroups (1), (2), (3) and (4). These crystals have, at most, one axis of degree $q$ normal to the plane of simple symmetry or alternative symmetry of degree $q$, where $q$ is any integer number. The magnetically polarized body has symmetry (d) $(L_\infty\ l_\infty)/P_\infty$ which is the limiting case of groups (5) and (6) for $q=\infty$. All crystals that, according to the theory, can have rotation coefficients have as a type of symmetry one of the subgroups of magnetic symmetry.

The theory built for crystalline bodies perfectly applies for magnetic symmetry, and the existence of rotational coefficients explains all the peculiarities of Hall's phenomenon, without the need to introduce into the theory of conductivity anything other than the symmetry of the field.

If the electricity is to come to the center of the metal disk located perpendicular to the magnetic field and if this electricity is collected uniformly on the edges of the disk, then the lines described by the electric flux must be spirals (Boltzmann)[9].

[9] It is very interesting that the crystals for which the theory of Stokes was created turned out to be refractory, whereas C. SORET unsuccessfully studied the impact of rotation coefficients in gypsum on thermal conductivity. The Hall phenomenon was observed only in the case of metals, and gypsum is a dielectric. Rotation coefficients would be perhaps perceptible upon using a crystallized metallic body exhibiting the necessary dissymmetry, but I do not think that we currently have the right substance to conduct an experiment.

The theory of thermal conductivity in crystals is presented in the latest article by C. SORET (Soret 1893, pp. 241-259). Lord KELVIN was the first to notice that Hall's phenomenon provided evidence of the existence of rotational coefficients (Thomson 1882).

### Pyroelectric and piezoelectric phenomena

Pyroelectric crystals by necessity have the symmetry of the subgroup of the electric field symmetry group, because the heating, by conjecture homogeneous, does not introduce any dissymmetry by itself. Piezoelectric crystals are more numerous than pyroelectric crystals. In fact, they include all these pyroelectric ones and also crystals that under the influence of mechanical load assume only the symmetry lesser than the symmetry of the electric field. For example, the blend (tetrahedral crystal) and quartz have symmetries that are not subgroups of the electric field. The quartz has symmetry $2L_3$, $3(L_2\ L'_2)$, a double 3-fold main axis and three 2-fold axes, non double perpendicular to this axis. For example, by squeezing along the 2-fold axis, cylindrical dissymmetry (a) is added to this of quartz; everything that remains as elements of symmetry is $(L_2\ L'_2)$, a 2-fold non double axis which can become the direction of electrical polarization.

It can also be demonstrated in the same way that by squeezing in the direction perpendicular to both the 2-fold axis and to the 3-fold axis, polarization will be created along the 2-fold axis and that the coefficients that affect the characteristics of these two modes of polarization generation are equal and have opposite signs. So, we can predict some special features of this phenomenon; but these symmetry conditions are not the only ones that occur in the general theory[10].

### 6. Relationships between characteristic symmetries of different media

We thought that the non-crystalline material with no rotational force does not introduce by itself whatever dissymmetry into the system; we adopted by default the same assumption for the medium which fills the empty spaces of the material. This quite natural, but completely heuristic assumption is necessary. It shows well that we cannot get the concept of absolute symmetry; we must arbitrarily choose symmetry for a specific medium and deduce the symmetry of other media. What is more, this relative symmetry is the only one that interests us. For example, if the whole system moves at a certain velocity and we consider in it a certain body A, then in general it will be useful to us to know the symmetry of body A relative to the system without taking into account the conjoined dissymmetry arising from the motion of the entire system.

Let us assume that in electricity we know only the general phenomena of static electricity, dynamic electricity, magnetism, electromagnetism and induction, then nothing will tell us exactly what kind of symmetry should be assigned to the electric field and the magnetic field. For example, for the magnetic field, we could choose symmetry (c) (which we assigned above to the electric field) and, reasoning as we did, we necessarily would have to take as symmetry of the electric field group (d) (which we assigned above to the magnetic field). In such a system, there would be no absurdity or contradiction with our initial hypothesis on total symmetry of matter.

General phenomena of electricity and magnetism, therefore, show us only the relationship between the symmetries of the electric field and the magnetic field, so that if we accept (c) for the symmetry of one, we must accept (d) for the symmetry of the other and vice versa. To remove this indeterminacy, it is necessary to introduce other phenomena, electrochemical

[10] The complete general theory of piezoelectric properties of crystals was developed by W. VOIGT (Voigt 1890; Riecke, Voigt 1892).

phenomena or contact electricity, pyro- or piezoelectric phenomena and even Hall phenomenon or magnetic rotational polarization.

The *dimensions* of electrical and magnetic quantities give an example of indeterminacy quite comparable to the one that we just cited for *symmetry* of electric and magnetic media. General phenomena of electricity and magnetism similarly are not able to remove this indeterminacy; to eliminate it, other phenomena should be taken into account, e.g. electrochemical phenomena[11].

### 7. Concluding remarks

The characteristic symmetries of phenomena are undeniably the subject of general interest. From the point of view of applications, we see that the conclusions that we can draw from the reflections on symmetry are of two types.

The first are some negative conclusions; they are a response to an undeniably true statement: *there is no effect without a cause*. The effects are the phenomena which always require some dissymmetry in order that they can occur. If this dissymmetry does not exist, then the phenomenon is impossible. This often stops us from wandering in search of phenomena impossible to realize.

Reflections on symmetry still allow us to formulate a second kind of conclusions, those of a positive nature, but which do not give the same certainty as those of a negative nature. They correspond to the statement: *there is no cause without any effects*. The effects are phenomena that may occur in a medium showing some dissymmetry; we have valuable hints here to discover new phenomena, but these predictions are not accurate predictions, such as those of thermodynamics. We have no idea about the order of the magnitude of the anticipated phenomena; we also only have an imperfect idea of their exact nature. This last remark shows that we must avoid drawing categorical conclusions from negative experience.

Consider, for example, a tourmaline crystal which has symmetry that is a subgroup of electric field symmetry. We come to the conclusion that such a crystal can be electrically polarized. Let us place the crystal in the electric field with the axis at 90° to the field. The polarization does not manifest itself in any way, there is no noticeable torque acting on the crystal and one could think that the crystal is not polarized or if the polarization exists, it is smaller than the one that could be measured. However, the polarization exists and for it to appear, the experiment should be modified, e.g. through homogeneous heating of the crystal which does not change anything in its symmetry.

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[11] An attempt in this direction has already been made by M. Abraham (Abraham 1893).

[12] Translator's note: In the original, P. Curie refers to the work of A. Bravais twice, the first time without giving specifics, the second time giving its title incorrectly. A literature search made it possible to identify the referenced work.

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[13] Translator's note: In this work, Fedorov was the first to derive 230 spatial (crystallographic) symmetry groups, now serving as the mathematical basis for structural analysis; see also Scholz 1989, pp. 114-148; Paufler, Filatov 2020; References for translator's Commentary.

[14] Translator's note: P. Curie did not give specific details of the referenced work; the most probably it is the indicated work.

[15] *Ibidem.*

[16] *Ibidem.*

[17] Translator's note: The original mistakenly gives the initial of the name as M. Soret. However, it is clear from the context and literature search that the work in question is that of Charles Soret, so the initial has been changed in the translation to C. Soret.

# Commentary to the English translation

Andrzej Ziółkowski

## 1. Footnotes to the main text

**P1**. The *definition of symmetry* outlined by Pierre Curie in several paragraphs below is neither clear nor precise. It can probably be best described as intuitive. On the one hand, it refers to the classic definition of symmetry, limited to geometric objects (figures, solids), having its source in ancient Greece. On the other hand, Pierre Curie creatively wipes the trail here to a very general contemporary understanding of the concept of symmetry as a universal property of comprehensive application. The work initiated by Pierre Curie was successfully completed by German mathematician Hermann Weyl who was the first to formulate the contemporary definition of the concept of symmetry as a certain universal philosophical category characterizing the organizational structure of all systems existing in the universe (Weyl 1952, p. 3):

> […] Starting with a slightly unclear concept of symmetry = harmony of proportions, in these four lectures gradually, first, it is developed the geometric concept of symmetry… to finally get to the general idea underlying all these special systems, namely the invariability of configuration of elements when subjecting them to a certain group of automorphic transformations. […]

According to Weyl, the quintessence of symmetry is property of the invariance of the object (of any kind) when subjecting it to a certain set (group) of automorphic transformations.

A general, very capacious *contemporary definition of symmetry* which the author of the present Commentary would formulate is as follows:

**Definition of Symmetry**

*Symmetry is the invariance* (stability, durability, constancy, isotropy) of some *feature* (geometric, physical, biological, informational, etc.) of an *object* (an object can be a geometric system, a material object, a natural phenomenon, a physical law, a social relation, a process running in time, a physical field, etc.) after subjecting it to *transformations from a certain set* (transformations can be shifts, mirror images, rotations, changes in order, etc.) with respect to which the symmetry is considered.

As shown by this definition, infinitely many different *types of symmetry* exist, depending on the category of objects considered, the type of object features analyzed and the types of transformations which the objects may be subjected to. With respect to some feature, an object may be simultaneously symmetric due to one type of transformation and dissymmetric due to another type of transformation. With respect to another feature, the same object may yet be symmetric relative to both previously considered types of transformations. The feature of symmetry is therefore a very comprehensive and rich concept.

A precise mathematical definition of symmetry can be found in Appendix B entitled *Symmetry*, in the book of Jan Rychlewski, *Dimensions and Similarity* (Rychlewski 1991, pp. 171–184). In the Appendix, a concise outline of the general *formal language of symmetry* (quantitative model of symmetry) is presented, applicable for the examination of any situation in which the concept of symmetry occurs. The key elements of the mathematical apparatus of *algebraic theory of symmetry* defined in Appendix B and discussed in examples are the concepts of: $\Gamma$ -sets, orbits, orbit markers, invariants and invariant functions. The algebraic theory of symmetry is a versatile tool enabling the analysis of all types of symmetry. In Appendix B, important results of the symmetry theory are briefly presented, such as the *ornament principle* (expressing the deepest property of complex symmetric objects in the simplest way), the *representative theorem* for symmetric objects, the theorem *on the symmetry of causes and effects of physical laws*, and the theorem on the invariant extension of any function.

The full mathematical theory of symmetry was developed in Jan Rychlewski's *Symmetry of causes and effects* (Rychlewski 1991) of which Appendix B is a very compact synopsis and motivation.

A mathematically precise contemporary definition of material symmetries, along with definitions of the necessary related concepts, is recalled in the Final Comments section of the present Commentary.

**P2**. In the light of the modern definition of symmetry, it can be guessed that by *recovery operations* the author understands, as for the essence, operations leaving a given system (object) unchanged, which are nowadays called *symmetry operations*. However, in all his work, the author uses the concept of recovery operations in a broader sense, namely, that of certain sets of operations of a special type (e.g. rotations, mirror reflections, etc.), and also subsets of specific operations that do not lead to a change (e.g. of shape) of a given object (e.g. rotations by 90$^o$), i.e. actual symmetry operations. For this reason, it was decided to use a literal translation of the original phrase, i.e. 'recovery operations', in the belief that this would prevent misunderstanding.

**P3**. From the broader context, it can be guessed that by displacement the author understands not only classic *linear* displacement, but also *angular* displacement, i.e. rotations. Thus, by *recovery (symmetry) operation of the first type* the author understands operation of the *linear* displacement or *angular* displacement, i.e. *rotation*. Also from the context, one can guess that by the *recovery operation of the second type* the author understands the *mirror reflection* operation. The author's definition of a mirror image operation as *'...symmetric transformation in the right sense...'* is, in the light of the modern definition of symmetry, inaccurate.

It is worth noting that the difficulties the reader may have with the correct understanding what types of symmetry are discussed originate apparently from a certain methodological error which is frequent even nowadays when defining various types of symmetry. Namely, transformations of a *symmetry element*, relative to which symmetry of the object is examined (e.g. inversion axis, alternating axis, mirror reflection of triad of the coordinate axes, etc.), are discussed rather than transformations of the *object* the symmetry of which is examined when submitting it to a specific type (group) of transformations. A substantively correct, contemporary definition of symmetry gives a hint on how to ensure precision and clarity in determining the type of symmetry under examination. Following it, one should talk about the inversion, mirror reflection, twisting or change of order of an object whose symmetry is being examined, and not about the inversion, mirror reflection, etc. of the symmetry element relative to which a given transformation operation possibly leading to symmetry is executed. Even if the latter is 'simpler' due to, for example, brevity of expression.

**P4**. As a standard, we imagine a rotational axis of symmetry as a straight line (line segment) around which the rotation is made. However, there are situatitons (shapes of objects) when the up-down orientation of the item relative to the rotation axis is important. A typical example is that of a pyramid with a polygon base mentioned by Pierre Curie. If the tip of the pyramid points up and the pyramid is rotated around its axis, then its shapes will coincide at some specific angles. Similarly, if the tip of the pyramid points down and the pyramid is rotated, then its shapes will also coincide at specific angles. However, at no rotation angle of the pyramid around its axis will the shape of the pyramid with the tip up coincide with the shape of the pyramid with the tip down. To be able to distinguish and describe such situations, the concept of *polar axes* was introduced, i.e., axes whose sense relative to the object examined in terms of its symmetry is important. Pierre Curie says that such axes are of a '*different type*' even though they actually concern the same axes and the same transformation types, e.g. rotations. The difference between axes $L_q$ and $l_q$ is graphically illustrated in Figure S1.

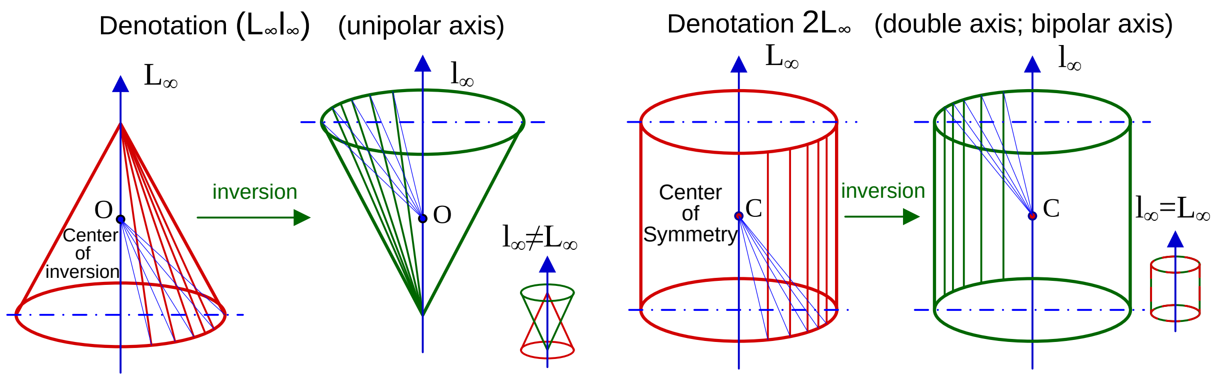


Fig. S1. Graphic illustration of the difference between axes of symmetry $L_q$ and $l_q$.

In crystallography, also today, many concepts of various 'types' of symmetry axes are used, e.g. inversion axis, alternating axis, roto-inversion axis, etc. The difference between the first two is graphically illustrated in Figure S2.

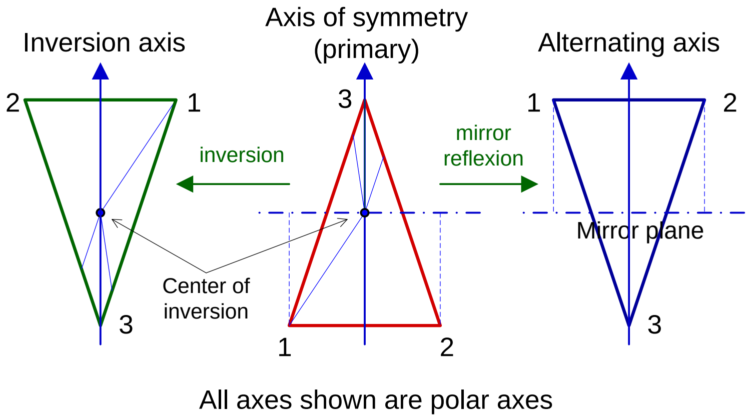


Fig. S2. Illustration of the principles of constructing the inversion and the alternating axes from the original (primary) axis of symmetry.

**P5**. The text often uses two names of the types of symmetry axes; these are the *2-fold axis* (*axe binaire*) and the *double axis* (*axe doublé*) which can be easily mistaken.

The *2-fold symmetry axis* means that when an object is rotated *around* it by $180^0$, the original shape and the shape after rotation coincide (the shape of the object is invariant). The *double* *axis of symmetry* means that when an object is turned upside down *relative to it*, rotated by $180^0$, the original shape and the shape after rotation coincide; see also Figure S1. Thus, completely different sets of recovery (symmetry) operations are associated with the 2-fold axis and the double axis. The literature often mentions that upon changing the sense of the axis, the object remains invariant. This expression is in principle incorrect, although illustrative, because it is the object whose symmetry is examined that is subjected to transformations in order to find out whether it is symmetric when subjecting it to a specific type of transformation.

It is also worth noting that the frequently used term *'repeatability' means rotation operation* around a certain axis of symmetry.

**P6**. One of the basic resources used in crystallography is the widely accepted Table of 32 *crystallographic classes* containing a list of all possible *symmetry point groups* of crystals (i.e. symmetries with respect to such *elements of symmetry* as *center of symmetry*, *plane of symmetry* and *symmetry axis*). The admissible *transformations* are three-dimensional rotations, inversions and mirror reflections; whereas in accordance with the *theorem on crystallographic restrictions*, crystals can only have 2, 3, 4 and 6-fold symmetry axes. Crystallographic classes were divided into seven *crystallographic systems*, with the division being based on sets of *symmetry groups* with one or more common elements of symmetry. Naturally, this is not the only possible classification and division. Presented in Pierre Curie's work, Table 1 makes a division of a certain set of objects analogous to the above- described standard division of crystals into *classes* and *families* with respect to *elements of symmetry* and *groups of symmetry* of these objects. However, the set of objects Curie considered is qualitatively broader than crystals (it contains physical fields), and the set of symmetry elements and the set of symmetry transformations considered are also broader.

By analogy, *Curie's crystallographic class* corresponds to the standard *crystallographic system*, while *Curie's family* corresponds to the standard *crystallographic class*.

The classic Table of 32 crystallographic classes concerns *static* geometric systems and *discrete* transformation operations that lead to invariance, i.e. in relation to which systems are symmetric (static crystallographic shapes are recovered through the discrete value of the angle of rotation around the axis of symmetry, etc.). Table 1. includes geometric systems in motion – *kinematic*, e.g. rotating, as is the case with a magnetic field or a sphere with vortical fluid. In Table 1., Curie takes into account *continuous transformations* that can lead to symmetry (invariance) of objects (rotation by any angle, even infinitely small, recovers shape). Continuous symmetry transformations lead to the concept of *limiting groups of symmetry*. The heuristic approach and non-standard elements contained in Table 1 proved to be creative and cognitively inspiring, and initiated the process of generalizing the concept of symmetry which led to the understanding of its deepest essence and developing a contemporary *definition of symmetry* formulated half a century later by Herman Weyl.

In order to highlight the relations between standard crystallographic systems and the symmetry classes distinguished by Pierre Curie, an extra column was added to Table 1. by the translator with specific international denotations of Curie's families, i.e. those of Hermann-Mauguin (H-M) classification. Table 3.2.1.3 (the 47 crystallographic face and point forms, names, eigensymmetries, and occurrence in the crystallographic point groups (generating point groups)), Figure 3.2.1.1 (the 47 crystal forms that crystals may take) and Table 3.2.1.4 (names and symbols of 32 crystal classes) presented in the work of Hahn, et al. (Hahn 2016) were very helpful in establishing these denotations.

**P7**. *Enantiomorphic* figures – from the Greek *enantios* meaning 'opposite'. Two objects, e.g. flat shapes and/or geometric solids, are enantiomorphic when they are mirror images of each other (are formed by a mirror reflection). As this definition implies, there can be only two objects (figures) which are mutually enantiomorphic. It also follows from the definition above that all enantiomorphic objects are congruent objects (they have the same size and shape).

The feature of enantiomorphism determines the way in which two enantiomorphic objects are formed, and thus how they are related to each other, but it does not determine whether or how they are symmetric. Enantiomorphic figures may or may not be symmetric due to some set of transformations other than mirror reflection. The feature of enantiomorphism does not specify (says nothing about) whether two flat enantiomorphic figures are right- and/or left-handed figures or whether two three-dimensional enantiomorphic objects (e.g. orthogonal triad) are right- and/or left-handed objects. Determination whether enantiomorphic objects are symmetric (superimposable on each other) due to rotation transformations requires further examination. In crystallography or mineralogy, two crystallographic systems may or may not be symmetric due to a set of rotations in three-dimensional space. In order to indicate that some enantiomorphic crystallographic systems are not symmetric due to rotations (they are not superimposable on each other due to some set/class of rotations), facets are often placed on the drawings of such systems to illustrate the type of dissymmetry.

Dissymmetric *enantiomorphic* objects (non superimposable on each other by rotation) are called *chiral* objects – from the Greek word *kheir* meaning 'hand'). The concept was proposed by Kelvin in 1894:

> '…I call any geometric figure or group of points, "chiral", and say that it has chirality, if its image in a plane mirror, ideally realized, cannot be brought to coincide by itself…', (Lord Kelvin, 1894).

The study of various types of chiral objects is currently a subject of very lively scientific interest, e.g., in pharmacological chemistry, since chiral molecules with the same chemical composition can exhibit radically different effects on the human body depending on their handedness (three-dimensional configuration of the internal structure). The contemporary definition of symmetry indicates that one should be careful when qualifying objects as chiral. For example, flat models of the left and right 'hands' obtained by mirroring them against a mirror plane set perpendicular to the model plane *are chiral due to two-dimensional rotations* limited to the model plane in which these models lie. By rotating them only in the model plane (two-dimensional rotations), they cannot be superimposed on each other to coincide. However, these models *are not chiral due to three-dimensional rotations*, e.g. transforming the 'left' model by rotating it by $180^0$ going outside the model plane allows this model to be superimposed on the 'right' model to coincide, so the models are then superimposable, cf. Figure S3.

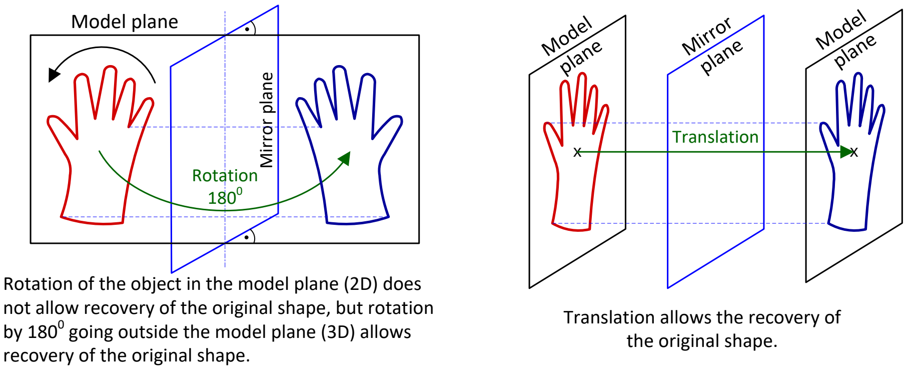


Fig. S3. Graphical illustration of the ambiguity of the definition of the chirality of an object, in the case of two-dimensional objects, due to the possibility of bringing the mirror image (Kelvin definition) into conformity with the original. Depending on the mutual alignment of the mirror plane and model plane and the class of admissible recovery (rotation) transformations, the same object can be qualified as chiral or not.

The mutual orientation of the mirror plane and the model plane is also important. When the mirror image plane is parallel to the model plane, then the flat model of the hand and its mirror image can be brought to coincidence by simple translation, so according to Kelvin's definition, such objects are not chiral.

Let us further consider a coordinate system in three-dimensional space (triad of versors). When the versors are indistinguishable, e.g. all are in black, then a person who has not seen how two triads were formed will not be able to tell their left- or right-handedness, and therefore determine whether they are chiral or not. This is because two orthogonal triads always can be superimposed on each other to coincide by rotation in three dimensions. However, when the individual versors are distinguishable, for example due to color differences, then it will be possible to determine whether two triads are chiral (one is right- and the other is left-handed), because it will be either possible to superimpose all versors by three-dimensional rotation so that they coincide in geometric position and colors (non-chiral triads) or not (chiral triads).

The examples considered above show that the deepest essence of chirality is not in the geometrical characteristics formulated by Lord Kelvin as its defining distinguishing feature, but in some *permutational (ordering) features of the internal structure* of an enantiomorphic pair. It seems to be natural to extend the concept of chirality to include the dissymmetry of the ordering of any two systems built with the same components. Take, for example, isomers of a molecule with four different types of ligands. We will have possibility of four (four!) factorial orders of the internal structure of such a molecule. In general, two isomers with a specific ordering of structure — in view of some property — can be dissymmetric or symmetric with respect to a certain pair of indices that characterize the internal structure, and therefore exhibit chirality or not. A similar situation occurs in the case of internal symmetry (with respect to a pair of indices) of the components of the fourth-order tensor.

The present discussion suggests the need and possibility of constructing a more precise and general *new definition of chirality*, in which the existence of *dissymmetry of a tensor characterizing the internal structure of an object*, e.g. with respect to a pair of specific indices, should be adopted as a defining indicator of chirality.

The above observation also provides a hint as to how a rational categorization and naming of chiral objects can be introduced using the concept of internal (permutational) symmetry of tensors.

The property of chirality and the various physico-chemical effects associated with it provide spectacular experimental evidence of the validity of the statement formulated by Pierre Curie that *it is dissymmetry that generates the occurrence of a phenomenon*.

**P8**. The equivalents between the contemporary international denotations of Hermann-Mauguin's (H-M) convention and Friedel's convention (using the concepts of *Holoedria*, *Hemiedria*, *Tetartoedria*, etc.) can be found, for example, in Table 3.2.1.4 of Hahn's et al. report (Hahn, 2016).

## 2. Final Comments

It is worth pointing out a few reference works that can facilitate understanding and help promote the use of the results of Pierre Curie's work for one's own needs.

*Free Textbook for College-Level Mineralogy Courses* (Anonim 2022) and *Mineralogy, Lecture Notes* (Nelson 2017) both provide clear and concise information on the current state of knowledge, contemporary crystallographic nomenclature, as well as graphic illustrative materials, e.g. three-dimensional models of various crystallographic shapes, photos of minerals and numerous other very helpful information.

Other very helpful descriptive and graphical explanations of terms and concepts relating to symmetry, including the concept of *limiting point groups* introduced by Pierre Curie, as well as non-standard, innovative kinematic elements defining the limiting point groups of symmetry, can be found in an article by A.V. Szubnikow (Szubnikow, 1956, English translation (1988)). Figures 1 and 2 from Szubnikow's article with a geometric interpretation and a schematic representation of limiting point groups were adapted in Figures S4 and S5. The following equivalence relations apply between the denotations of limiting axial groups of symmetry introduced by Pierre Curie in Section 4 of the translated work and the nominal Hermann-Mauguin (H-M) international denotations: $(a) \leftrightarrow \infty / mm$, $(b) \leftrightarrow \infty 2$, $(c) \leftrightarrow \infty m$, $(d) \leftrightarrow \infty / m$, $(e) \leftrightarrow \infty$. The symmetry of the electric field is $\infty m$ (a stationary cone), the symmetry of the magnetic field is $\infty / m$ (a rotating cylinder). The symmetry $\infty\infty m$ denotes isotropy (a stationary sphere).

All limiting symmetry groups contain the same *common element of symmetry*, i.e. the axis of symmetry of an infinite degree.

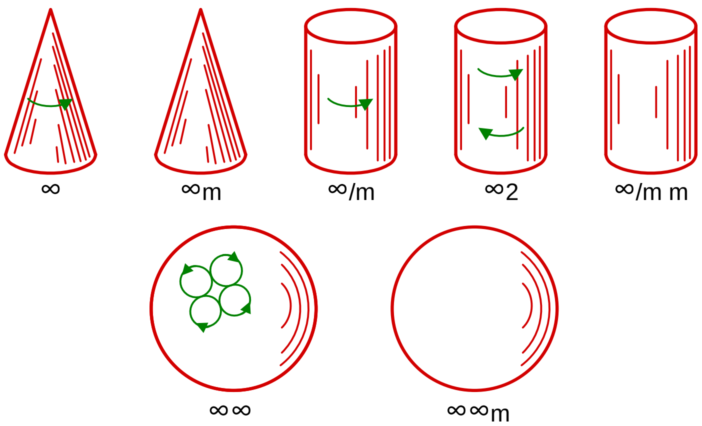


Fig. S4. Geometric interpretation of the limiting point groups of symmetry; an adaptation of Figure 1 from the work of Szubnikow (Szubnikow, 1956, English translation (1988)).

The *elements of symmetry* of limiting symmetry groups are as follows: the group of a *rotating cone* $(\infty)$ has a symmetry axis of an infinite degree, the group of a *stationary cone* $(\infty m)$ has a symmetry axis of an infinite degree and an infinite number of planes of symmetry containing the axis of symmetry, the group of a *rotating cylinder* $(\infty / m)$ has a symmetry axis of an infinite degree, one transverse plane of symmetry and a center of symmetry, the group of a *twisted cylinder* $(\infty 2)$ has a symmetry axis of an infinite degree and an infinite number of transverse 2-fold axes of symmetry, the group of a *stationary cylinder* $(\infty / m\, m)$ has a symmetry axis of an infinite degree, an infinite number of transverse and longitudinal planes of symmetry, an infinite number of transverse 2-fold axes of symmetry and a center of symmetry, the group of a *sphere with no symmetry planes and no center of symmetry* $(\infty\infty)$ has an infinite number of symmetry axes of an infinite degree; it is a sphere with all diameters twisted to the right or left; the group of a *stationary sphere* $(\infty\infty m)$ has an infinite number of symmetry axes of an infinite degree, an infinite number of symmetry planes, and a center of symmetry.

It is worth noting that attempts to describe the symmetry of crystallographic systems using second-order symmetric tensors, as encountered in crystallographic literature, cannot be successful in the light of current knowledge.

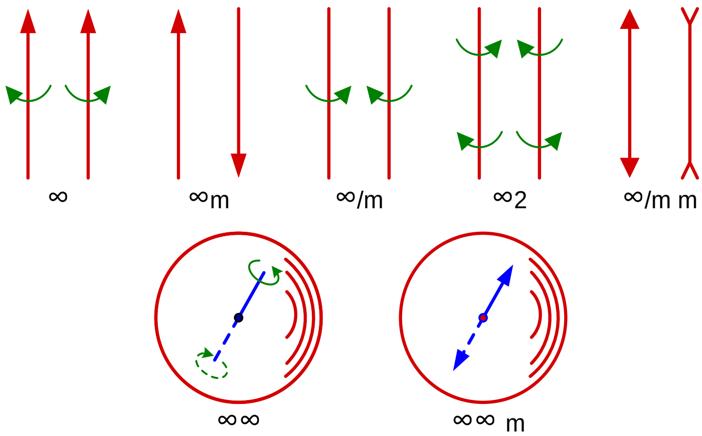


Fig. S5. Schematic graphical illustration of limiting point groups of symmetry; adaptation of Figure 2 from the work of Szubnikow (Szubnikow, 1956, English translation (1988)).

A correct tensorial description of the symmetry (anisotropy) classes of elastic properties of three-dimensional (3D) materials considered in the mechanics of continuous media, i.e. the *external symmetries* of Hooke's tensor with respect to the group of orthogonal tensors $\boldsymbol{Q}$ $(\boldsymbol{Q}^T\boldsymbol{Q} = \mathbf{I})$ describing rotations and mirror reflections, can be found in the work of (Kowalczyk-Gajewska, Ostrowska-Maciejewska 2009). Equivalence (correspondence) relations between the classes of symmetry of crystallographic systems and the classes of symmetry of linear-elastic materials (Hooke's tensor symmetry classes) can be identified, see Table P1.

The material symmetries of elastic properties of two-dimensional (2D) materials can be correctly described using second-order symmetric tensors. Such a description can be found in the work of (Blinowski et al. 1996).

Sandra Forte and Maurizio Vianelllo (Forte, Vianello 1996) proved in 1996 that there exists a maximum of 8 classes of symmetry for a linearly elastic material (Hooke's tensor). It should be remembered that Forte and Vianelllo's findings are valid if and only if the tensor describing the material properties — here: the Hooke's tensor — has the following *internal symmetries*:

$\mathbf{H}^{<1234>} = \mathbf{H}^{<2143>} = \mathbf{H}^{<3412>}$ $(\sim H_{ijkl} = H_{jikl} = H_{klij})$ $i, j, k, l = 1, 2, 3$, where $H_{ijkl}$ denote the components of the Hooke's tensor in any fixed tensorial basis.

**Table P1.** Equivalence relations between classes of symmetry of *crystallographic systems* and classes of symmetry of *elastic properties of linearly elastic materials* (external symmetries of the Hooke's tensor).

| **No.** | **Cl. of symmetry of crystallo-graphic system** | **No.** | **Class of material symmetry** | **Notes** |
|---|---|---|---|---|
| **1.** | triclinic | **1.** | anisotropy | |
| **2.** | monoclinic | **2.** | monoclinic | |
| **3.** | orthorhombic | **3.** | orthotropic | |
| **4.** | tetragonal | **4.** | tetragonal | |
| **5.** | trigonal (rhombohedral) | **5.** | trigonal | In recent studies, the symmetries of trigonal and hexagonal systems are increasingly adopted as a single crystallo-graphic system (hexagonal). |
| **6.** | hexagonal | | – | |
| | – | **6.** | transversely isotropic | Transversely isotropic (cylindrical) symmetry is, in the nomenclature introduced by Pierre Curie, the limiting symmetry of rhombohedral system. The classical systematics of crystallographic systems does not contain this symmetry. |
| **7.** | cubic | **7.** | cubic | |
| | – | **8.** | isotropy | Isotropy is the limiting symmetry of a cubic (regular) system, absent in the classical systematics of crystallographic systems. |

The material model, e.g. the Hooke's tensor, should not be confused with a real body, e.g. crystalline material. The Hooke's tensor is a model of linear-elastic behavior and its symmetries reflect the symmetries of such behavior. However, the Hooke's tensor is used to model the behavior of crystalline materials and, for example, amorphous materials, as long as the behavior of a given material in a certain range of loadings can be considered as linear and elastic with a good approximation. The division into crystallographic systems is based on identification of certain common elements of symmetry characterizing the internal structure of crystalline materials. Nothing prevents the symmetries of the Hooke's tensor from coinciding to some extent with the symmetries of crystallographic systems. However, it can be expected that the symmetries of the Hooke's tensor will be broader (richer) than the symmetries of crystallographic systems (from the point of view of a linear-elastic behavior), because the Hooke's tensor generally enables a description of a wider range of materials than just crystalline materials. Reality confirms this, because crystallographic systems do not include Curie's *limiting symmetries*, while the symmetries of the Hooke's tensor do.

### 3. Mathematical definitions of symmetries used in materials research

Below are recalled some mathematically precise, modern definitions of internal symmetries (due to permutations of indices) and of external symmetries (due to rotations and mirror reflections in three-dimensional space) of a set of tensors of order $p$, along with definitions of related concepts. These types of symmetries are currently most often used to characterize the symmetry of material properties in their tensorial description. More information on this subject can be found in Janina Ostrowska-Maciejewska's book (Ostrowska-Maciejewska 2007).

***Definition S1.*** A *permutation operation* $\sigma \times$ *on tensor* $\mathbf{T}$ is a linear mapping defined with the following rule

$$\begin{aligned}&\sigma\times\mathbf{T}:\ \mathbf{T}=T_{12\ldots p}\,\mathbf{e}_1\otimes\mathbf{e}_2\otimes\ldots\otimes\mathbf{e}_p\rightarrow\sigma\times\mathbf{T}=T_{12\ldots p}\,\mathbf{e}_{\sigma(1)}\otimes\mathbf{e}_{\sigma(2)}\otimes\ldots\otimes\mathbf{e}_{\sigma(p)},\\&\sigma\equiv<\sigma(1),\sigma(2),\ldots,\sigma(p)>,\quad \mathbf{T},\sigma\times\mathbf{T}\in\mathcal{T}^p,\end{aligned}\tag{S.1}$$

where $\sigma(1),\sigma(2),\ldots,\sigma(p)$ is a preset permutation of the first $p$ natural numbers $1,\ldots,p$, and $T_{1,2,\ldots,p}$ are components of tensor $\mathbf{T}$ of $p$-th order in the tensorial basis $\mathbf{e}_1\otimes\mathbf{e}_2\otimes\ldots\otimes\mathbf{e}_p$. A permutation of a tensor means change in the order of components of its tensorial basis.

The permutation operation can be interpreted, in a completely equivalent manner, as a permutation of the components of the tensor representation written down in a fixed basis,

$$\sigma\times\mathbf{T}\equiv<\sigma(1),\sigma(2),\ldots,\sigma(p)>\times\mathbf{T}=T_{\sigma(1)\,\sigma(2)\ldots\,\sigma(p)}\,\mathbf{e}_1\otimes\mathbf{e}_2\otimes\ldots\otimes\mathbf{e}_p\in\mathcal{T}^p,\tag{S.2}$$

For permutation operations $\sigma$ of a tensor, it is convenient to introduce the following more compact notation $\sigma\times\mathbf{T}\equiv<\sigma(1),\ldots,\sigma(p)>\times\mathbf{T}\equiv\mathbf{T}^{<(\sigma(1)\,\sigma(2)\ldots\,\sigma(p)>}$. When it is known from the context that the order of only two indices changes, it is convenient to specify only those indices that are changed, e.g. in the case of fourth-order tensors $\mathbf{T}^{<4,2>}$ instead of $\mathbf{T}^{<1,4,3,2>}$.

The permutation operation is an automorphism, i.e. it is a reversible, linear transformation of tensor space $\mathcal{T}^p$ on itself $(\sigma:\ \mathcal{T}^p\xrightarrow{on}\mathcal{T}^p)$.

The set of all permutation transformations operating in the space of tensors of a fixed order constitutes the group $(\mathcal{P}^\sigma)$, cf. Definition S9, which allows introducing the concept of the *internal symmetry of tensors*. The size of this group is finite and equals $p!$ elements, for example, for tensors of the 4-th order there are $4!=24$ elements in this group.

***Definition S2.*** An *internal symmetry group* of tensor $\mathbf{T}\in\mathcal{T}^p$ is a subset of the permutation group $\mathcal{P}^\sigma$, whose elements satisfy the condition

$$\mathcal{P}^\sigma_{\mathbf{T}}\equiv\{\sigma\in\mathcal{P}^\sigma;\ \ \sigma\times\mathbf{T}=\mathbf{T}\},\quad \mathcal{P}^\sigma_{\mathbf{T}}\subset\mathcal{P}^\sigma.\tag{S.3}$$

The tensors $\mathbf{T}$ satisfying the condition (S.3) are called (*internally*) *symmetric* tensors with respect to permutations $\sigma\in\mathcal{P}^\sigma_{\mathbf{T}}$.

A tensor $\mathbf{T}$ is *(internally) symmetric* over a pair of indices $(\alpha,\beta)$, if equality holds, $\mathbf{T}=\mathbf{T}^{<\beta,\alpha>}, \sim T_{\ldots\alpha\ldots\ldots\beta\ldots}=T_{\ldots\beta\ldots\ldots\alpha\ldots}$, i.e. when the elements of the tensor $\mathbf{T}$ representation in any fixed basis when swapping the places of indices $(\alpha,\beta)$ are the same. In the case of fourth-order tensors, the symmetry with respect to permutation operation $<1,3,2,4>\times$ means that $\mathbf{T}=\mathbf{T}^{<1,2,3,4>}=\mathbf{T}^{<1,3,2,4>}$, i.e. $T_{ijkl}\to T_{ikjl}$ in any fixed basis.

***Definition S3.*** A tensor is *absolutely (internally) symmetric* when the group of its symmetries is the entire set of permutations $\mathcal{P}_{\mathbf{T}}^{\sigma}=\mathcal{P}^{\sigma}$.

***Definition S4.*** A set of second-order tensors $\mathbf{Q}$ with properties,

$$\mathcal{O}=\{\mathbf{Q}\in\mathcal{T}^2;\ \mathbf{Q}\mathbf{Q}^T=\mathbf{Q}^T\mathbf{Q}=\mathbf{1},\quad \det\mathbf{Q}=\pm1\} \qquad \text{(S.4)}$$

is a group and is called the *group of orthogonal tensors*.

***Definition S5.*** A subset of orthogonal tensors for which

$$\mathcal{R}=\{\mathbf{Q}\in\mathcal{T}^2;\ \mathbf{Q}\mathbf{Q}^T=\mathbf{1},\ \det(\mathbf{Q})=+1\},\quad \mathcal{R}\subset\mathcal{O} \qquad \text{(S.5)}$$

is a group and is called a *proper (special) orthogonal group* or *rotational group*. In the literature, this group is often denoted by the symbol $SO_3$, in the case of a three-dimensional Euclidean space generating the considered tensor space.

***Definition S6.*** An *external symmetry group* of tensor $\mathbf{T}\in\mathcal{T}^p$ is the subset of all orthogonal tensors $\mathbf{Q}$ that satisfy the following condition

$$\mathcal{O}_{\mathbf{T}}=\{\mathbf{Q}\in\mathcal{O};\ \ \mathbf{Q}*\mathbf{T}=\mathbf{T}\},\quad \mathcal{O}_{\mathbf{T}}\subseteq\mathcal{O}\ \ (\mathbf{Q}*\mathbf{T}\leftrightarrow Q_{ia}Q_{jb}...Q_{kc}T_{ab...c})\ . \text{(S.6)}$$

Tensors $\mathbf{T}$ satisfying the condition (S.6) are called *(externally) symmetric* with respect to orthogonal transformations $\mathbf{Q}\in\mathcal{O}_T$.

***Definition S7.*** A tensor is *isotropic* when its group of external symmetry is the whole set of orthogonal tensors $\mathcal{O}_T=\mathcal{O}$, cf. (S.4).

***Definition S8.*** A tensor is *hemitropic* (also called *proper-isotropic*) when its external symmetry group is the entire set of *proper orthogonal* tensors $\mathcal{O}_T=\mathcal{R}$, cf. (S.5).

***Note.*** The above definitions clearly show that the symmetry property is a property of a tensor treated as an integrated entity composed of a basis and a representation (a matrix of components in a given basis), and not only a matrix of tensor components.

**Example.** If a fourth-order tensor has three internal symmetries $\mathbf{T}\equiv\mathbf{T}^{<1,2,3,4>}=\mathbf{T}^{<2,1,3,4>}$, $\mathbf{T}=\mathbf{T}^{<1,2,4,3>}$, $\mathbf{T}=\mathbf{T}^{<3,4,1,2>}$ and all its eigenvalues are non-negative, then it is called a Hooke's tensor. The tensor is used to describe (model) the elastic properties of a linearly elastic material. The external symmetries of the Hooke's tensor, that is, its invariance when subjected to the operation of orthogonal tensors $\mathbf{Q}$ from certain subsets determine the symmetries of the material modeled with its help.

The concept of a group is one of the most important concepts widely used in building theories (models) of real physical phenomena.

***Definition S9.*** A *group* is an *algebraic structure* $G\equiv(\{G\},\Diamond)$ consisting of a non-empty set of *elements* $\{G\}$, and an operation $"\Diamond"$ that assigns an element from $\{G\}$ to any pair of elements from $\{G\}$ $(\Diamond:\ (g,h)\in\{G\}\times\{G\}\Rightarrow g\Diamond h\in\{G\})$, when the operation $\Diamond$ satisfies the following axioms

$$(i) \bigwedge_{g_1,g_2,g_3 \in G} \quad g_1 \lozenge (g_2 \lozenge g_3) = (g_1 \lozenge g_2) \lozenge g_3,$$
$$(ii) \bigvee_{e \in G} \bigwedge_{g \in G} \quad e \lozenge g = g \lozenge e = g, \quad (iii) \bigwedge_{g \in G} \bigvee_{h \in G} \quad g \lozenge h = h \lozenge g = e \qquad \text{(S.7)}$$

i.e. the operation $\lozenge$ is *associative (i),* there exists a *neutral element* of the group (*ii*), for each element of the group there exists an *inverse element* (*iii*).

A group is called commutative (*Abelian Group*) when the operation $\lozenge$ is commutative

$$(iv) \bigwedge_{g,h \in G} \quad g \lozenge h = h \lozenge g .$$

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