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Ladislas Natanson and Alfred Landé versus Planck's law, the Boltzmann-Planck-Natanson statistics and the Bose statistics

Abstract

The article describes the context and content of the November 1925 correspondence – so far overlooked by historians of physics – between Władysław (Ladislas) Natanson and Alfred Landé on Planck's law and Bose statistics, and the effects of this interaction.

The article publishes for the first time the transcription of two original letters in German and their translations into English.

Keywords: *Władysław Natanson, Ladislas Natanson, Alfred Landé, Boltzmann's statistical method, Boltzmann statistics, Planck's law, Planck statistics, Bose statistics, Boltzmann-Planck-Natanson statistics, Boltzmann-Planck-Natanson-Bose statistics.*

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Władysław Natanson i Alfred Landé a prawo Plancka, statystyka Boltzmanna-Plancka-Natansona oraz statystyka Bosego

Abstrakt

Artykuł opisuje kontekst i treść przeoczonej dotąd przez historyków fizyki korespondencji z listopada 1925 roku na temat prawa Plancka i statystyki Bosego pomiędzy Władysławem Natansonem, a Alfredem Landé i konsekwencje tej korespondencji.

W artykule publikowane są po raz pierwszy transkrypcje dwóch oryginalnych listów w języku niemieckim i ich tłumaczenia na język angielski.

Słowa kluczowe: *Władysław Natanson, Ladislas Natanson, Alfred Landé, prawo Plancka, metoda statystyczna Boltzmanna, statystyka Boltzmanna, statystyka Boltzmanna-Plancka-Natansona, statystyka Bosego, statystyka Boltzmanna-Plancka-Natansona-Bosego.*

1. Introduction

So far, in the works of historians of physics, it is impossible to find information whether Władysław Natanson ever reacted to the publication of two articles by Satyendra Nath Bose (1924a; 1924b) on Planck's law or Planck statistics.[1]

[1] See e.g. Whithaker 1953; Jammer 1966, p. 51; Hund 1967 (translations: 1974, 1975, 1980); Mehra 1975; Średniawa 1985, pp. 89–90; 2000, pp. 454–455; 1996/2006; Lange 1992a; 1992b, p. 132; 1997; Wali 2006; Konieczny 2011; 2012; Nagasawa 2018, p. 412; Saunders 2020; Spałek 2005a, p. 15 / 2005b, p. 152; 2020; Badino 2021.

This issue has not been analyzed in the currently most extensive work on the reception of Natanson's achievements concerning the so-called Bose-Einstein statistics (Kokowski 2019). It mentions, however, that on November 18, 1925, Alfred Landé sent a letter from Tübingen to Władysław Natanson (Landé 1925 (*archival document*). Walther Gerlach and Alfred Landé in 1926 quoted Natanson's article from 1911 in the German version (Kokowski 2019, p. 379) and that Natanson himself, in his article "O promieniowaniu" ("On radiation") (2nd version) published in the book *Oblicze natury: odczyty, przemówienia i szkice* (*The Face of Nature: Readings, Speeches and Sketches*) (pp. 125–151) was silent about Bose's article for a prosaic reason, namely because this

On the other hand, it is known from such studies that Alfred Landé (1888–1976) played a role in explaining the new description of electromagnetic radiation in 1925–1926. He did this through four publications on light quanta (Landé 1925; 1926a; 1926b; Landé, Gerlach 1926) and his correspondence with Erwin Schrödinger, about which Albert Einstein was also informed.[2]

Fig. 1. Władysław Natanson, circa 1910. Source: Majkowska, Fiałek (eds.) 2009, illustration 14.

Fig 2. Alfred Landé ca. 1940 (by unknown author – Aus dem Privatbesitz der Familie Landé, Public Domain). Source: https://commons.wikimedia.org/w/index.php?curid=17990743.

The purpose of this article, on the basis of the preserved sources and their interpretation in the light of the methodology of the history

book had appeared before the publication of Bose's article (Kokowski 2019, p. 342). After additional research, the article author was able to establish beyond any doubt that Natason's book was published before April 14, 1924, because on April 14, 1924, it had already been reviewed in the *Wiadomości Literackie* journal (see Wasowski 1924, p. 3). Moreover, since it is known that Bose sent his article to *Zeitschrift für Physik* only on June 2, 1924 and did not correspond with Natanson, it is highly probable or almost certain that Natanson did not know about the existence of Bose's article at the time when Natanson's book was published.

[2] See e.g. Mehra, Rechenberg 1987, pp. 417–419.

of physics,[3] is to show that Władysław Natanson at least once openly reacted to the aforementioned Bose's articles, and this is closely related to the above-mentioned works by Alfred Landé and Walther Gerlach.

2. Landé's article from August 1925

On June 19, 1925, Landé sent an article to the *Zeitschrift für Physik* entitled "Lichtquanten und Kohärenz" ("Quants of light and coherence"),[4] which was published in August that year (vol. 33, pp. 571–578).[5]

In an abstract to the article, Landé outlined the content of this work:

> According to Bose, if one wants to derive Planck's radiation formula from the light quantum theory, it can only be done with the help of probability approach that regards the light quanta as entities statistically dependent on one another in an unexplained way and attributes polarization to them. However, this probability approach can also be justified in the case of statistical independence of the light quanta, if, in analogy to classical wave interference, the scalar addition of the quanta ε within each quantum bundle is abandoned in favor of a superposition of the $\sqrt{\varepsilon}$ with randomly

[3] See Appendix 1.

[4] Historians of physics usually skip this 1925 article by Landé. They did not pay attention to him e.g. Whithaker 1953; Jammer 1966; Hund 1967 (translations: 1974, 1975, 1980); Mehra 1975; Kuhn 1978; Kastler 1983; Średniawa 1985; 2000; 1996/2006; Bergia 1987; Lange 1992a; 1992b; 1997; Stachel 2002; Wali 2006; Hentschel 2006; 2018; Waniek, Hetschel 2011; Konieczny 2011; 2012; Passon, Grebe-Ellis 2017; Nagasawa 2018; Saunders 2020; Spałek 2005a, s. 15 / 2005b, s. 152; 2020; Badino 2021.

The few who noted this article, e.g. Raman, Forman 1969, p. 313, fn. 66; Hanle 1977, p. 186; Wessels 1979, p. 322, fn. 51 & 52; Rechenberg 1982; Mehra, Rechenberg 2001, p. 418; Kojevnikov 2002, p. 202; Lehner, Renn, Joas, Badino 2007, pp. 6–7, 9, 12; Fick, Kant 2013, pp. 109, focused at best on the idea of returning to the wave theory of light and Lande's quantum interference, and failed to see any connection between this and Natanson's publication (1911a / 1911c). Such behavior has its justification, because Erwin Schrödinger did not notice this relationship in the letter of November 3, 1925 to Albert Einstein (Einstein 2018, doc. 101, pp. 182–183) and in the letter of November 16, 1925 to Alfred Landé (Schrödinger 1925 (*archival document*) and Walther Bothe (1927), although Landé's publication was mentioned in each of these cases.

[5] From the perspective of the editor of a scientific journal, I must emphasize that this is a very short period of time for conducting a reliable review procedure and a thorough editorial preparation of the text.

> distributed phases, while only allowing integer quantum bundle energies. The relationship between this interference approach and Einstein's theory of gas degeneration are pointed out. The quantum bundles also prove to be fundamental for the exchange of energy between radiation and matter (translation – M. Kokowski).[6]

Landé developed these threads in the following parts of the article. In the first subsection, he recalled the quantum derivation of the factor $8\pi\upsilon^2/c^3$, in the formula for the density of energy states of blackbody radiation by Bose (Bose 1924a, pp. 179–180; Landé 1925a, pp. 571–572). Then he considered the specific case and determined the number of possible distributions of the *four* light quanta in *three* phase cells of the size h^3, as determined by Planck.[7] The number of these equally probable distributions is 15 (Landé 1925a, p. 573, table 1).[8] Here they are:

Table 1.

Cell No. 1	4	0	0	3	3	1	0	1	0	2	2	0	2	1	1
Cell No. 2	0	4	0	1	0	3	3	0	1	2	0	2	1	2	1
Cell No. 3	0	0	4	0	1	0	1	3	3	0	2	2	1	1	2

[6] "Will man die Plancksche Strahlungsformel aus der Lichtquantentheorie ableiten, so gelingt dies nach Bose nur mit Hilfe eines Wahrscheinlichkeitsansatzes, der die Lichtquanten als voneinander in unaufgeklärter Weise statistisch abhängige Gebilde ansieht und ihnen Polarisation zuschreibt. Dieser Wahrscheinlichkeitsansatz kann aber auch bei statistischer Unabhängigkeit, der Lichtquanten begründet werden, wenn man, in Analogie zur klassischen Welleninterferenz, die skalare Addition der Quanten ε innerhalb jedes Quantenbündels aufgibt zugunsten einer Superposition der $\sqrt{\varepsilon}$ mit nach Zufall verteilten Phasen, unter Zulassung nur ganzzahliger Quantenbündelenergien. Es wird auf die Beziehungen dieses Interferenzansatzes zu Einsteins Theorie der Gasentartung hingewiesen. Die Quantenbündel erweisen sich ferner als grundlegend auch für den Energieaustausch zwischen Strahlung und Materie" (Landé 1925a, p. 571).

[7] Planck (1906) considered the number of possible different configurations of ΔN quanta of energy in ΔA oscillators; Debye (1910) – ΔN energy quanta in ΔA natural vibrations; while Laue (1914) – ΔN energy quanta in independent beams of rays.

[8] Such a result (i.e. 15 cases) was given earlier by Planck (1906, pp. 152–153), who gave four-digit numbers such that each digit is 1 or 2 or 3 if the energy element is in resonator 1 or 2 or 3. Such a result was also given by Natanson, using a simpler notation of these cases (1911a, pp. 137–138, § 3; 1911c, pp. 660–661, § 3; 1913, pp. 58–61, § 42). Natanson's record was also repeated by Landé (without reference to Natanson).

Commenting on these results, Landé pointed out that light quanta could nevertheless be *a priori* independent, and thus distinguishable:

> It is by no means self-evident that these 15 possibilities are equally likely ("have the same weight"); on the contrary, assuming mutually independent quanta, one would evaluate a priori on the assumption of mutually independent quanta that the distribution (3, 1, 0) would have a weight four times greater than the distribution (4, 0, 0), because the latter can only be realized in one way, in that all four quanta a, b, c, d fall into cell no. 1 (abcd, 0, 0), whereas the former can be realized in four ways: (abc, d, 0), (abd, c, 0), (acd, b, 0), (bcd, a, 0) (transl. – M. Kokowski).[9]

Hence:

> The equal probability of the above 15 distributions from four quanta to three cells is therefore to be regarded as a special implicit physical assumption, which is based on Bose's calculations and whose content is to be looked for in the dependence of quanta in the sense of a certain interference ability [...] (transl. – M. Kokowski).[10]

In other words, the equal probability of such 15 configurations means that Bose and also Planck before that, which was not mentioned by Landé, made an implicit physical assumption in their models, the indistinctness of light quanta of the same energy.

According to Landé, the cause of this effect is the "coherence of light quanta" (1925a, p. 571) or the "certain interference ability"

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[9] "Es ist keineswegs selbstverständlich, daß diese 15 Möglichkeiten gleich wahrscheinlich sind („gleiches Gewicht haben"); vielmehr würde man a priori unter der Annahme voneinander unabhängiger Quanten erwarten, daß der Verteilung (3,1,0) ein viermal so großes Gewicht zukäme als der Verteilung (4,0, 0), weil letztere nur auf eine Weise realisierbar ist, indem alle vier Quanten *a, b, c, d* in Zelle Nr. 1 fallen: (abcd,0,0), erstere dagegen auf vier Weisen: (abc, d, 0), (abd, c, 0), (acd, b, 0), (bcd, a, 0)" (Landé 1925a, p. 573).

[10] "Es ist die Gleichwahrscheinlichkeit der obigen 15 Verteilungen von vier Quanten auf [drei Zellen demnach als eine besondere physikalische Annahme anzusehen, die implizite Boses Rechnungen zugrunde liegt und deren Inhalt in einer Abhängigkeit der Quanten im Sinne einer gewissen] interferenzfähigkeit zu suchen ist […]" (Landé 1925a, p. 573).

of light quanta (1925a, p. 573), or "interference of light quanta" (1925a, p. 574). It is manifested in the Bose derivation of the thermodynamic probability of a macroscopically defined light quantum distribution, which uses the formula for the number of permutations with repetitions (Bose 1924a, pp. 180–181 / English version: Bose 1976, p. 1057).

According to Landé, in a general case, the number of possible configurations (ΔW) of ΔN quanta in ΔA cells according to Bose's calculation scheme is given by the formula:

$$\Delta W = \frac{(\Delta N + \Delta A - 1)!}{(\Delta A - 1)!\,\Delta N!}$$ (Landé 1925a, p. 573, formula 2),

which is the number of "combinations with ΔA repeats of elements with ΔN classes" (Landé 1925a, p. 573, formula 2).

In claiming this, Landé overlooked to mention in this context that Boltzmann had already provided this formula (1877/1909, p. 181), as well as Planck (1900, p. 240; 1906, p. 152) and Natanson (1911a, p. 136; 1911c, p. 660), who quoted Boltzmann and Planck, while Bose himself did not provide this formula in his publications (1924a; 1924b).

According to Landé, this led Planck to deduce the formula:

$$\frac{\Delta E}{\Delta A} = \frac{\varepsilon}{e^{\varepsilon/kT} - 1}$$ (Landé 1925a, p. 573, formula 3).

Then Landé – following the example of (unnamed) Boltzmann and Natanson – derived the Planck distribution from the assumption of the indistinctness of light quanta and the distinguishability of cells.[11] The starting point was to determine the number of different configurations

[11] See Klein 1962, pp. 472–474 (comparison of Boltzmann approach and Planck approach); Kuhn 1978, pp. 102–110 (description of Planck's approach); Pais 1979, pp. 893–895 (comparison of Boltzmann's and Bose's approaches); 1982, pp. 370, 423––428; 1986, p. 283 (comparison of Planck's and Bose's approaches); Mehra, Rechenberg 1982/2001, pp. 566–569 (comparison of Planck's and Bose's approaches); Bergia 1987, pp. 223–227 / 2009, pp. 333–337 (comparison of Boltzmann's and Bose's approaches); Bach 1988 (comparison of, among others, Boltzmann's, Planck's, Natanson's and Bose's approaches); Bach 1990 (comparison of, among others, the Boltzmann's approach, Planck's approach and Natanson's approach); Monaldi 2009, pp. 387–388; 2019, pp. 319–320, 323–324 (comparison of Boltzmann's approach, Planck's approach and Natanson's approach); Enders 2016 (comparison of Boltzmann's approach and Planck's approach); Saunders 2020, pp. 48–49 (comparison of Boltzmann's approach and Natanson's approach).

Δw N light quanta in ΔA cells, which is determined by the polynomial distribution for "occupancy numbers of groups" instead of the occupancy numbers themselves, it is also the permutation of the number ΔA of cells with repetitions of the numbers of indistinguishable light quanta in cells j = 0, 1, 2,… Boltzmann, for example, used a similar formula (1877/1909b, p. 176, formula 3; p. 187), see also Planck (1906, p. 152)[12]:

$$\Delta w = \frac{(\Delta A)!}{(p_0 \Delta A)!\,(p_1 \Delta A)!(p_2 \Delta A)!\ldots}$$ (Landé 1925a, p. 574, formula 8),

where p_j and $p_j \Delta A$ – probability of finding j-light quanta in cell ΔA and the number of indistinguishable light quanta in cells j = 0,1, 2, ….

In the next step, Landé made two elementary assumptions for the occupancy numbers, known at least from Boltzmann (1877/1909, pp. 175–176, formulas 1 and 2, p. 205), see Landé 1925a, p. 575, formula 9:

$$\sum_j p_j = 1, \quad \sum_j p_j \varepsilon_j = \frac{\Delta E}{\Delta A}$$

and used the known elementary condition for the log Δw to reach a maximum. Consequently, he set the value:

$$p_j = Be^{-j\varepsilon\beta}$$ (Landé 1925a, p. 575, formula 10).

Additionally, because:

$$\sum_j p_j = 1, \quad B = 1 - e^{-\varepsilon\beta},$$

$$p_j = (1 - e^{-\varepsilon\beta})e^{-j\varepsilon\beta}$$ (Landé 1925a, p. 575, formula 10').

Hence the average energy ū per cell is:

$$\bar{u} = \frac{\Delta E}{\Delta A} = \sum_j \varepsilon_j p_j = \frac{\varepsilon}{e^{\varepsilon\beta} - 1}$$ (Landé 1925a, p. 575, formula 11).

[12] In Boltzmann (1877/1909b, p. 176), it is the so-called permutability of energy distribution Π = n! / (w0! w1! ...), ie the number of possible configurations, compatible complexes, for the energy of the E system. Since Boltzmann's time, this formula has played an important role in both classical and quantum statistical physics. The sum of the permutabilities is J = (λ+n–1)! / (N–1)! λ!) (Boltzmann 1877/1909b, p. 181) has the same form as Landé ΔW (see the formula on the previous page). Planck (1906b, p. 152) referred to it as *Wahrscheinlichkeit* (probability) and marked it with the symbol W; while according to Boltzmann W = Π / J. Natanson (1911a, pp. 136–138; 1911c, pp. 660–661) knew about these issues; see also note 20 (below) and Darrigol 1991, especially pp. 244–250, 257, 293.

Due to formula 3, β = 1 / kT and hence Landé finally got:

$$p_j = (1 - e^{-\varepsilon/\mathrm{kT}}) e^{-j\varepsilon/\mathrm{kT}}$$ (Landé 1925a, p. 575, formula 10").

For the limit case of $\varepsilon \ll kT$, Landé finally got:

$$p_j = \frac{\varepsilon}{\mathrm{kT}} e^{-j\varepsilon/\mathrm{kT}}$$ (Landé 1925a, p. 575),

and after inserting ū = kT:

$$p_j = \frac{\varepsilon}{\bar{u}} e^{-j\varepsilon/\bar{u}}$$ (Landé 1925a, p. 575, formula 12).

Landé additionally believed that each quantum of light should be treated as a wave and attributed it with not only a polarity and amplitude ($\sqrt{\varepsilon}$), but also a phase (φ): $\sqrt{\varepsilon} \cdot e^{i\varphi}$ and as a vector quantity (not a scalar), hence the superposition of light quanta is a vector composition (Landé 1925a, pp. 575–577). Moreover, according to Landé, an analogous effect of "molecular interference or superposition" also exists in Einstein's theory of gases (Landé 1925a, pp. 577–578).[13]

3. Forgotten letter from the Archive for the History of Quantum Physics, 1898–1950.

In the Archive for the History of Quantum Physics, 1898–1950, kept at the American Philosophical Society Library (Philadelphia, PA, USA), there is a letter from Władysław Natanson to Alfred Landé, forgotten by historians of physics, sent on November 14, 1925 from Kraków to Tübingen (it is located in "Volume reel 4: reel-frame 19, Call Number Mss.530.1.Ar2").

Natanson wrote a letter in German to Landé after reading his article from August 1925 (Landé 1925a), which reached Kraków only in November 1925. In this letter he:

[13] Landé repeated the same considerations also in the monograph *Die neuere Entwicklung der Quantentheorie* [Newer development of the quant theory] (1926b, pp. 13–19). Landé's return to wave interpretation (vector quantum superposition) in the derivation of Planck's law aroused the interest of Erwin Schrödinger – see two letters from October 28, 1925 and November 16, 1925: Schrödinger 1925a (arch. doc.); 1925c (arch. doc.) And footnote 15 (below).

a) referred to the theses proclaimed by Landé;
b) recalled the fact that in March 1911 he had published an article in English in the *Biulletin International de L'Academie des Sciences de Cracovie. Classe des Sciences Mathématiques et Naturelles Série A; Sciences Mathématiques* (Natanson 1911a, pp. 134–148),[14] and in August 1911 its German translation entitled "Über die statistische Theorie der Strahlung" [On the statistical theory of radiation] in *Physikalischen Zeitschrift* published in Leipzig (Natanson 1911c, vol. XII, pp. 659–666), in which he analyzed in detail the question of quantum indistinctness in the derivation of Planck's law, and discussed the same or analogous examples as Landé did, pointing to the relevant pages from the English version of the article (Natanson 1911a).[15]

Moreover, according to Natanson, Landé's original idea was the idea of a random vectorial superposition of light quanta, but that idea needs to be further developed based on the work of Lord Rayleigh (1871/1899a; 1880/1899b; 1884, § 42a; 1888/1902; 1899/1903).[16]

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[14] The title of this article is Polish-English "O teoryi statystycznej promieniowania. – On the Statistical Theory of Radiation", although these were the only Polish words in the entire article written in English.

[15] For the German original of the letter, a transcription of the letter and an English translation – see Appendix 2; for a Polish translation see Kokowski 2021, Dodatek 2.

[16] Other critical remarks on this idea were made by Erwin Schrödinger in a letter to Albert Einstein of November 3, 1925 (see Einstein 2018, doc. 101, pp. 182–183; Mehra, Rechenberg 1987, p. 418) and of November 16, 1925 to Landé (see Raman, Forman 1969, p. 313).

Schrödinger, Erwin 1925b (*archival document*) original letter (p. 182), English translation (p. 121): "Currently I am exchanging letters with Landé about his quantum interferences [Zeitschrift] f[ür] Phys[ik], 33, p. 571) (Lande 1925). The idea seems to me to be very interesting but not properly thought through. What is Planck's (instead of Wien's) radiation law supposed to result in if the individual quanta are distributed throughout the cells, as if they did the interference independently and only retroactively, inside their cells, but only such that the mean energy of a larger number of cells, each containing j quanta (of the same kind), is not altered by the interference? Landé's response is: Well, it is not the energy that is altered, but the entropy level that is supposed to be judged from the resulting content of the individual cells. Very well. But the entropy here only plays the role of a measure of probability. Consequently, if the probability is supposed to be judged from the resulting cell content, well, then the quanta will not be distributed quasi-independently throughout the cells and the whole advantage of L's conceptual approach, which, of course, is supposed to be an

Natanson added two more important comments in this letter, the first concerning the priority problem – Natanson saw it as a trivial problem not worth spending time on, and the second concerning the understanding of the idea of indistinguishability of quantum particles – according to Natanson, readers of his article did not understand the gap in Planck's probabilistic approach.[17]

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explanation for the strange preference of quanta to "squat together," […] seems to me to be lost.–Two other objections are aimed at the faulty application of the formulas (which only apply with large numbers) to very small numbers, and are less meaningful in principle than the first objection, if it is valid."

Einstein first used the expressions "Boseschen Statistik" ("Bose statistics") and "klassischen Statistik" ("classical statistics") in a letter to Schrödinger of February 28, 1925 – see Einstein 1925 (*archival document*) – German original (Einstein 2015a, Doc. 446); English translation (2015b, Doc. 446); this fact was noted by Daniela Monaldi (2019, p. 328).

[17] Natanson was right about the indistinguishability of quantum particles, because it was assumed to be treated on the grounds of quantum mechanics as a postulate that is not explained further Natanson was right in this matter, because the indistinguishability of quantum particles was assumed to be treated on the grounds of quantum mechanics as a further inexplicable postulate – see e.g. Dirac 1926, pp. 662, 667–673; 1927, p. 12; 1930, pp. 198, 201, 208, 219–220; Heisenberg 1926, pp. 422–424; 1927; Wigner 1927a; 1927b; Pauli 1940, p. 718. In the mentioned article by Dirac (1926, p. 673), its author incorrectly derives the formula for the distribution of the energy of molecules: the formula differed in sign from the Bose-Einstein theory based on Bose-Einstein statistics, which Dirac referred to as the Einstein-Bose theory; this mistake was later reproduced by G. E. Uhlenbeck and P. Ehrenfest 1926 (*archival document*), as noted by Einstein 1926 (*archival document*).

See also Sudarshan, Mehra 1970, on the classical statistical mechanics of identical particles and quantum effects; Mehra, Rechenberg 1982/2001, on the relationship between the symmetry of the wave function and quantum statistics, pp. 757–768 (explanation of Dirac's error: pp. 765–766); Mehra 2001, pp. 622–633, explanation of Dirac's error: pp. 632–633; Fraassen 1998, especially pp. 75–76, postulate of invariance of permutations, pp. 79–82, identity of bosons; pp. 82–86, theoretical unification of various statistics; pp. 86–88, causality and correlations.

Currently, researchers see the limitations of the accepted view of identical particles in quantum mechanics – see e.g. Dieks 2020; Spalek 2020. This fact is not surprising, however, in the light of the existence of a limit area of a theory defined by the limit of a corresponding parameter in the context of the hypothetico-deductive method of corresponding-oriented thinking (M. Kokowski's terminology).

4. Commentary from the perspective of the methodology of the history of physics

In the articles mentioned above, Landé analyzed Bose's model and Natanson – Planck's model;[18] while in a letter to Landé, Natanson pointed out that the combinatorial and thermodynamic properties of Bose's model, including the explanation of the problem of *indistinguishability of light quanta*, had already been recognized in Planck's model, which was what Natanson did in his article (1911a / 1911c), in which he showed that we can derive Planck distribution of the blackbody radiation energy:

> unless we take for granted that, in the process of estimating probabilities, the receptacles of energy *can* be treated as distinguishable and that the energy-units, being in all respects precisely alike, *cannot* be so treated (Natanson 1911a, p. 139).[19]

> wenn wir voraussetzen, daß bei dem Prozeß der Wahrscheinlichkeitsberechnung die Energiehalter als identifizierbar behandelt werden können, und daß die Energieeinheiten, die in jeder Hinsicht vollkommen gleich sind, nicht als identifizierbar behandelt werden können (Natanson 1911c, p. 602).[20]

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[18] In an epistemological sense, every mathematico-physical model is a mathematico-physical theory. However, it is customary in physics to reserve the term "model" for not fully developed theories – see Bailer-Jones 2009, pp. 1–46; 81–105.

[19] Klaus Hentschel (2018, p. 2012), who – contrary to the facts – believed that Natanson's article in *Biulletin International de L'Academie des Sciences de Cracovie* was published in Polish (in fact it was published in English), gave his own English translation of this quote from the German version of Natanson's article: "if during the process of calculating the probability we presume that it is possible to treat the receptacles of energy as identifiable and that it is not possible to treat the units of energy, which are the same in every respect, as identifiable" (Hentschel 2018, p. 82).

[20] According to the author of the article, there is no doubt that before Ehrenfest, Natanson understood the principle of the indistinctness of "energy elements" (understood as calculus fictions or empirical entities – see Appendix 4. This is evidenced by: a) an article by Natanson in English (read at a faculty meeting on March 6, 1911; published around April 10, 1911) and in German (received: April 29, 1911; published: August 15, 1911); b) Ehrenfest's notes from the spring of 1911 (this observation was overlooked by Navarro, Pérez 2004, pp. 118–124).

Three things need to be made clear here:

1) In his analyses of Planck's law, Natanson followed Boltzmann's footsteps.[21]

Ehrenfest knew Natanson's work: Ehrenfest himself spoke about it in his letter to Sommerfeld of October 16, 1911 (see Kokowski 2019, pp. 370–371, fn. 66); G. Krutkow, a student of Ehrenfest quoted it in his works (1914a; 1914b) and Krutkow's works were also quoted by Ehrenfest himself (Ehrenfest 1911; 1925/2016, p. 3, fn. 6; Ehrenfest, Kamerlingh-Onnes 1915a / 1915b) and recommended them to other physicists: he learned about them this way, for example, Hendrik Antoon Lorentz – see Kokowski 2019, pp. 372–374, fn. 69–74. Nevertheless, Ehrenfest – which is reprehensible – consistently omitted Natanson's articles in his publications (Ehrenfest 1911; Ehrenfest, Kamerlingh-Onnes 1915a / 1915b; Ehrenfest 1925/2016, p. 3, note 6), although they depended on them.

Planck, during the First Solvay Congress, which was held in Brussels from October 30 to November 3 in 1911 (Planck 1912, p. 104/1958, vol. 2, p. 277), responded positively to the existence of ambiguities in the distribution of energy elements and therefore changed the derivation of the formula for the distribution of energy: "This calculation is completely inambiguous, and nothing is left of the indetermination recently pointed out by L. Natanson" (cited by Darrigol 1991, p. 254). The reason for Planck's conclusion was his belief in the definition of Boltzmann's complexions, which was to remove this ambiguity (cf. Darrigol 1988, p. 52). I would add that Planck was wrong: the new derivation was still flawed (cf. Darrigol 1991, pp. 253–254).

Albert Einstein neither quoted Natanson's publications nor discussed them in his correspondence – see Einstein 1916; 1917; 1924; 1925; 1930; Einstein, Ehrenfest 1923; Einstein 1987–2018; Pérez, Sauer 2010. Nevertheless, Einstein had come across these publications by Natanson on at least two occasions: for the first time during or after the First Solvay Congress in 1911, as it was quoted by Planck 1912, p. 104, fn. 1 – see Kokowski 2019, p. 372; for the second time from reading the article by Krutkov (1914a), mentioned in the letter by Otto Halpern to Einstein of August 26, 1924 (Halpern 1924 (*archival document*), criticizing the derivation of Planck's radiation law from Bose's assumption on the independence of light quanta and citing the articles by Natanson (1911c) and Ehrenfest (1911) – see Kokowski 2019, p. 378, fn. 86.

[21] Natanson (1911a, pp. 134–138; 1911c, pp. 659–661; 1913, pp. 54–61) using Boltzmann's formula (1868; 1877) presented descriptions of the analysis of the statistical system configuration. Alexander Bach explained in detail and in many aspects the related issues in many important publications – see Bach, Blank, and Francke 1985; Bach 1985; 1987; 1988a; 1988b; 1990a; 1990b; 1990c; 1991; 1997, pp. 137–139; see also Enders 2016, pp. 2–3.

The starting point is to distinguish three levels of analysis of the statistical system.

The basic description is defined by random configuration variables: a system composed of N atoms endowed with energy elements / energy units; a specific atom is endowed with specific elements / units of energy; this is the Natansonian "energy

2) In an article from 1911, Natanson did not talk about light quanta ("Lichtquanta") – a term introduced in 1905 by Einstein,[22] but about energy-units ("Energieelemente") – a term introduced in 1900 to analyze the law of radiation of light by Planck in reference to Boltzmann (1872).[23] Natanson did not originally define the ontic status of the energy elements. On the basis of Natanson's reference to Boltzmann (1872, p. 275 / 1909a, p. 316), it could be assumed, however, that they were mathematical fictions. Yet, in later publications from 1911–1924, Natanson explained in detail that they were not mathematical fictions, although he sometimes expressed some doubts as to this conclusion – see Appendix 4.

3) In a letter to Landé of November 14, 1925, Natanson raised a new theme: he indicated the corresponding elements of these models: the

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distribution", "mode of association" (German: *Energieverteilung*; Polish: *rozmieszczenie energii*).

The second level is determined by the *occupation numbers* of random variables: the number of energy elements / units in individual (numbered) atoms: this is the Natansonian "energy distribution", "mode of collocation" (German: *Anordnungsart*; Polish: *rozkład energii*); one constraint (conditional equation) per total number of energy units.

The third description level is determined by the *occupancy numbers* of random variables: the numbers of atoms that have consecutive numbers of energy elements / units (0, 1, 2,…); this is the Natansonian "energy distribution", "mode of distribution" (German: *Verteilungsart*; Polish: *rozdział energii*) with two constraints (conditional equations) for the number of atoms and for the total number of energy elements / units.

For the case when we can distinguish individual atoms, but we cannot distinguish energy units, the probability of distribution (depending on the *occupancy numbers*) is determined by the Boltzmann probability measure $P=[(N-1)!n!N!]/[(N+n-1)!N_0!N_1!\ldots N_p!)]$, while for the case when we can distinguish both individual atoms and energy units, measure $P = [N!n!]/[N^n N_0!N_1!\ldots N_p!\ 0!^{N0}\ 1!^{N1} \ldots\ p!^{Np}]$. The rationale for the choice of the Boltzmann measure is its relationship with entropy (Natanson 1911a, pp. 134–138; 1911c, pp. 659–661; 1913, pp. 54–61).

[22] See *Einstein* 1905*; Wikipedia* 2021.

[23] Planck 1900, p. 239: "The distribution of energy over each type of resonator must now be considered, first, the distribution of the energy E over the N resonators with frequency ν. If E is regarded as infinitely divisible, an infinite number of different distributions is possible. We, however, consider – and this is the essential point – E to be composed of a determinate number of equal finite parts and employ in their determination the natural constant $h= 6.55 \times 10^{-27}$ erg sec. This constant, multiplied by the frequency, ν, of the resonator yields the energy element ε in ergs, and dividing E by hν, we obtain the number P, of energy elements to be distributed over the N resonators" (cited by Kuhn 1978, pp. 104–105).

"phase space cell" is the equivalent of the "energy carrier" called by him the "receptacle of energy"("Energiehälter"), and the "energy element" or "unit of energy", which is identified with the material unit possessing discrete energy values,[24] i.e. the quantum of light.[25]

Bose and Einstein overlooked the issue of the indistinguishability of quantum objects in 1924–1925, *because they did not understand the subtleties of Natanson's 1911 considerations at the time*.[26]

However, contrary to the argument of Friedrich Hund in 1967,[27] Natanson never stated that he "formulated the Bose statistics of light quanta before Bose",[28] because a) he did not provide in 1911 a quantum derivation of the density of energy states, which Bose presented in 1924[29] and b) in the years 1911–1923 he had some doubts about the reality of light quanta, which he, however, significantly reduced in 1924–1925 (see Appendix 4). However, it is still true thesis that

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[24] According to Saunders (2020, p. 48) Natansonian elements of energy are only mathematical fictions and nowhere has he identified them with quanta: "He nowhere explicitly identified his energy-units as light quanta. They were abstract from the beginning (he did not so much as mention the concept of frequency)". These are wrong theses. Natanson did not comment on this in his synthetically written papers on Planck's radiation theory (1911a / 1919c), but elaborated on these issues in his other publications – see Appendix 4.

[25] Previous researchers, not knowing Natanson's letter to Landé, made independent interpretations of the relationships linking Natanson's and Bose's considerations, i.e. the Boltzmann-Planck-Natanson statistics with the Bose statistics. These include the analyzes made by researchers such as: Artur Kastler (1983, p. 616); John Stachel (2000/2002, pp. 438–439); Silvio Bergia (1987, p. 344); Jagdish Mehra and Helmut Rechenberg (2001, p. 559); Daniela Monaldi (2009, 2019); Simon Saunders (2020, p. 49) and Alexander Bach (see works cited in footnote 16). These interpretations are critically discussed in Appendix 5.

[26] Cf. Kojevnikov 2002, pp. 201–202; Pérez, Sauer 2010, pp. 12–16; Kokowski 2019, p. 381.

[27] Cf. Friedrich Hund (1967, pp. 25–26, 134, 153–154; English translation: Hund 1974, pp. 30, 145; Russian translation: Hund 1980, pp. 26, 123; "Die Abzählung, die Natanson verdeutlicht hat, ist genau die, die Bose später auf Lichtquanten anwandte und die man jetzt Bose-Statistik nennt" (Hund 1967, p. 26); "This method of counting events that Natanson made is exactly the one that Bose later made for light quanta, and is now called Bose statistics" (Hund 1974, p. 30) – cited in: Kokowski 2019, p. 348.

[28] By statistics, the author of the article understands here a statistical model of phenomena.

[29] See Kokowski 2019, pp. 395–396.

Natanson was the first to argue that in order to derive Planck's law, one must adopt the principle of indistinguishability of quantum objects (*actually* units of energy, energy elements or light quanta, see Appendix 4). They proclaimed this thesis in one form or another, among others, Max Jammer (1966, p. 51), Friedrich Hund (1967/1974; 1975; 1980) and Artur Kestler (1983) and others[30].

5. A forgotten letter from Alfred Landé to Władysław Natanson

In volume 10 of Władysław Natanson's correspondence, collected at the Jagiellonian Library and available online, there is a letter from Alfred Landé to Władysław Natanson, sent on November 18, 1925 from Tübingen to Kraków in response to Natanson's letter of November 14, 1925, which was so far overlooked by historians of physics.

In this letter written in German, Landé: a) emphasized that, unfortunately, he had not previously known Natanson's 1911 article (although the article was published in English in *Biulletin International de l'Academie des Sciences de Cracovie* and in German at *Physikalischen Zeitschrift* – MK), b) expressed appreciation for Natanson's fundamental accomplishments, revealing, as he put it, the ambiguity of Planck's theory unnoticed by hundreds of other theorist physicists, and c) made a promise to quote Natanson's article at the earliest opportunity.
The English translation of this letter follows:

Tübingen, 18 November 1925
Dear Colleague!

Thank you very much for your letter and the reference to your beautiful investigation of 1911. I have just read the work and regret that it had not become known to me earlier. Your clear and critical, down to the last one, thoughtful considerations are of fundamental importance to any quantum theorist, and it is curious enough that none of many hundreds of theorists apart from you, have grasped the uncertainties of Planck's theory. Therefore, I am eager to seize the next opportunity

[30] See Kokowski 2019, pp. 348–357.

to point out your important and fundamental work. Some offprints will be sent at the same time to you.

Respectfully yours,
A. Landé[31]

6. An article by Gerlach and Landé (1926) and a monograph by Landé (1926, 2nd ed.)

On January 27, 1926, Walther Gerlach and Alfred Landé sent an article to the *Zeitschrift für Physik* entitled "Ein Experiment über Kohärenzfähigkeit von Licht" [Experiment on the coherence of light], which had been already published in March 1926 (vol. 34, pp. 169–173). In this article, the authors mentioned Natanson's 1911 article in the following context:

> From the standpoint of the most extreme light quantum theory, the interference cannot be understood at all[1)]. *But Planck's law of radiation in its Rayleigh-Jeans sub-area already leads to a rejection of the extreme light quantum conception*[2)]. However, a softening of the light quantum theory provides advice here, namely the requirement that the light quanta that belong to the same elementary bundle do not combine their energies additively, but rather by means of superposition[3)] after they have been assigned phases after polarization. With this additional assumption, however, the interference phenomena are not yet sufficient; for the parts of a ray of light can also interfere if the ray of light is so violet or so weak that it belongs to Wienn's spectral range. In the latter case, however, most of the elementary light quanta bundles are empty, and only in exceptional cases does a bundle carry a single light quantum with it, and occupancy with several light quanta is completely negligible. The ability of Wienn's light to interfere thus requires superposition of light quantum fractions" (translation, italic – M.K.).

[31] For the German original of the letter and a transcription of the letter see Appendix 2; for a Polish translation see Kokowski 2021, chap. 5.

[1] On the failure of the previous light quantum theories of interference cf. e.g. fn. 2, p. 322 in A. Landé, ZS. f. Phys. 85, 317, 1926]. [i.e. Landé 1926a].
[2] L. Natanson, Phys. ZS. 12, 659, 1911 [i.e. Natanson 1911c].
[3] A. Landé, ZS. f. Phys. 88, 571, 1925 [i.e. Landé 1925].[32]

In the same very concise way Landé quoted Natanson's work in another publication, namely in the bibliography of the 2nd ed. of the monograph *Die neuere Entwicklung der Quantentheorie* [The recent development of quantum theory, 1926b, p. 169, no. 45].

Thus, Landé fulfilled his promise to Natanson in a letter on November 18, 1925, but he did so in a very perfunctory and enigmatic manner, significantly different from the content of the letter sent to Natanson.[33]

7. Conclusions

In this article, the following irrefutable important facts were established:

a) in 1926 the well-known physicists Walther Gerlach and Alfred Landé mentioned at least twice in their publications on the property of light the article by Natanson from 1911 (in German version: 1911c);

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[32] "Vom Standpunkt der extremsten Lichtquantenlehre aus ist die Interferenz bisher gar nicht zu verstehen[1)]. *Aber auch das Plancksche Strahlungsgesetz in seinem Rayleigh-Jeansschen Teilgebiet kommt bereits zu einer Ablehnung der extremen Lichtquantenauffassung*[2)]. Jedoch schafft hier eine Milderung der Lichtquantentheorie Rat, nämlich die Forderung, daß die Lichtquanten, welche zum gleichen Elementarbündel gehören, ihre Energien nicht additiv zusammensetzen, sondern superponieren[3)], nachdem man ihnen Phasen und Polarisation zugeschrieben hat. Durch diese Zusatzannahme ist aber den Interferenzerscheinungen noch nicht genügt; denn die Teile eines Lichtstrahls können auch dann interferieren, wenn der Lichtstrahl so violett oder so sehwach ist, daß er zum Wienschen Spektralbereich gehört. In letzterem Falle sind aber die meisten elementaren Lichtquantenbündel leer, and nur ausnahmsweise führt ein Bündel ein einziges Lichtquant mit sich, und Belegung mit mehreren Lichtquanten ist ganz zu vernachlassigen. Die Interferenzfähigkeit Wienschen Lichtes verlangt also Superposition von Lichtquantenbruchteilen" (Gerlach, Lande 1926, p. 170; italics – M.K.).

[1)] Über das Versagen der bisherigen Lichtquantentheorien der Interferenz vgl. z. B. Fußnote 2, S. 322 bei A. Landé, ZS. f. Phys. 85, 317, 1926].

[2)] L. Natanson, Phys. ZS. 12, 659, 1911.

[3)] A. Landé, ZS. f. Phys. 88, 571, 1925.

[33] The author of the article is not aware of any archival documents that would explain the reason for such a decision and does not speculate about this decision.

b) it happened as a result of an exchange of two letters in German between Natanson and Landé on November 14, 1925 and November 18, 1925 (first published here together with translations into English and Polish).

As Natanson pointed out in his letter sent on November 14, 1925, Landé's derivation of Planck's distribution is analogous to the method presented by Natanson in 1911 and based on Boltzmann's approach: using the *occupancy numbers* of random variables of a configuration of a statistical system composed of N atoms endowed with energy elements / energy units.

Although in 1926 an article by Natanson from 1911 (in the German version) was quoted twice by Walther Gerlach and Alfred Landé, it did not arouse – to my knowledge – interest in scientific publications and was forgotten until the publication of three monographs from the history of physics: Edmund Taylor Whithaker (1953), Max Jammer (1966, p. 51) and especially Friedrich Hund (1967/1974, 1975, 1980).[34]

[34] See respectively Kokowski 2019, pp. 399–400 (about Whithaker) and pp. 346––347 (about Hunt). In this article, Jammer's remarks about Natanson were overlooked. According to Jammer, at the same time, two researchers Ehrenfest (1911) and Natanson (1911a / 1911c) independently noticed that Einstein's non-interacting quanta hypothesis led not to Planck's law of radiation, but to Wienn's law. Jammer was the first to summarize what Natanson's combinatorial approach was:

"In his analysis of the precise assumptions underlying Planck's combinatorial procedure in which P energy elements ε were distributed among N «receptacles of energy» so that N_j receptacles each contain *j* energy elements, subject to the restrictions $\Sigma N_j = N$ and $\Sigma jN_j = P$, Natanson called the correlation of jε with N_j, that is, sorting the receptacles according to their energy content and specifying the number of receptacles which have equal energy content, a «mode of distribution,» in setting up a «mode of distribution,» Natanson emphasized, no account is taken of a possible «identifiability» or distinguishability, as we would say today, of receptacles or of energy elements. However, as soon as the former are regarded as individually identifiable, every given «mode of distribution» ramifies into a number of «modes of collocation» which specifies the number of energy elements in each individual receptacle. Finally, if the energy elements are also considered as identifiable, each «mode of collocation» splits into a number of «modes of association» which associates individual energy elements with individual receptacles. Natanson then pointed out that the thermodynamic probability of a given «mode of distribution» depends notably on whether all «modes of association» or all «modes of collocation» are regarded as equally probable, and he showed that Planck, in contrast to Einstein, adopted the latter alternative" (Jammer 1966, p. 51).

Appendix 1.

Methodologico-sociological introduction

Each mathematical-physical theory is created by the collective effort of the scientific community as a result of complex interactions involving the analysis of mathematical-physical issues, the art of scientific research, scientific styles and thinking styles (Fleck, Crombie) and the art of argumentation, including styles of reasoning (Hacking), and the art of persuasion.

These interactions led to the emergence of the general method of exact sciences already in ancient times, which in the terminology of the article author called *the hypothetico-deductive method of correspondence-oriented thinking*.[35] The application of this method determines the mathematical-physical style of scientific research along with the style of thinking and the style of reasoning.

Thanks to the application of this method, mathematical and physical theories of phenomena are created. The postulated theories are constructed in such a way as to be internally consistent and to "save phenomena", i.e. that the predictions of the theory are consistent with empirical measurements.[36] In the course of the development of science, theories are constantly transformed and generalized (evolutionary-revolutionary mechanisms are manifested here), thus preserving the memory of earlier theories (their theoretical-empirical components).

In the process of reinterpretation and generalization of existing theories, which is constantly repeated in the history of mathematico-physical sciences, an important role is played by:

Jammer made a strong conclusion: "*It was in Natanson's analysis of Planck's statistical procedure that the problem of the distinguishability of elementary entities was raised for the first time*" (tamże; kursywa – M.K.).

However, Max Jammer (1966) was wrong about the simultaneity of the achievements of Ehrenfest and Natanson, Natanson has priority here – see above note 15.

[35] Ever since Thomas S. Kuhn (1962) and Paul Feyerabend (1975), philosophers of science and sociologists of scientific knowledge have denied the existence of the scientific method. This is a serious misunderstanding – see Kokowski 1996, pp. 10–25; 2004, pp. 59–62; 2012; 2015.

[36] These are known issues of the history and philosophy of mathematico-physical sciences, derived from the tradition of Plato (*Timajos*), Ptolemy (*Almagest*) and Copernicus (*Commentariolus*, *De revolutionibus*) – see Kokowski 2004, pp. 92–95, 157–162 – and also of Einstein – see Holton 1979; Howard, Giovanelli 2019.

a) correspondence parameter,
b) postulate of the theory correspondence,
c) the principle of correspondence of the theory,
d) a thought experiment with a change in the value of the correspondence parameter;
e) the theory boundary defined by the corresponding parameter boundary (usually 0 or ∞).[37]

In each mathematico-physical theory, three semiotic layers can be distinguished:

- mathematical layer: a specific mathematical language (this layer defines the syntax of the theory);
- quasi-entity layer: hypothetical physical entities that serve to explain phenomena; they are treated either as accounting fictions or empirical entities, which depends on the layer of empirical theory indicated below (this layer together with the empirical layer determines the semantics of the theory);
- empirical layer: determined by correspondence rules that either connect the new theory with old theories or directly lead to the measurement of the parameters of the new theory (this layer together with the quasi-entity layer determines the semantics of the theory).[38]

These layers do not need to be uniform at all. With the help of detailed historical research – a specific archeology of scientific knowledge – we can reveal in the theories under study the existence of hidden semiotic sub-layers inherited from previous theories.

In particular, the naming conventions of laws, rules and theories used are often very illusory, hiding the actual co-authors of such discoveries. To remedy this, first of all, remember about this dangerous feature of the convention and, secondly, supplement the list of the most important co-authors of such discoveries (Kokowski 2019, p. 398).

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[37] See Kokowski 1996, pp. 10–25; 2004, pp. 59–62; 2012; 2015.

[38] *Ibidem*; moreover see Badino 2015a; 2016 (considerations on the structure of theory and the process of reconfiguration of methods, research questions and epistemological frameworks in the historiography of quantum mechanics); Hentschel 2006; 2018 (considerations on the history, mental models and semantic layers of the concept of "light quantum"); Hacking 1982; 1990; 1992a; 1992b; 2002; 2012; Gavroglu 1990; Pickering 1992; Elwick 2012; Monaldi 2019 (considerations on the statistical style of reasoning; research practices and cultures; layers of history).

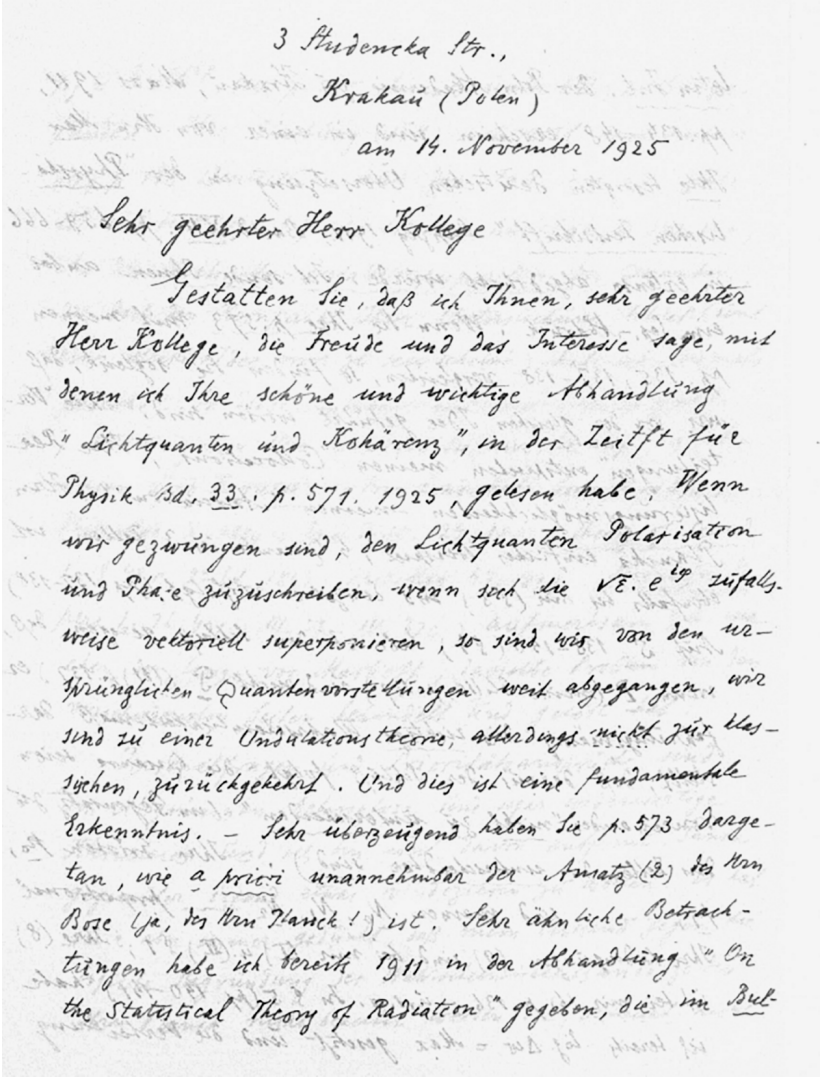
Nevertheless, any conventional name of this type always remains just a convention, and as such does not reveal the full complement of the actual contributors to the findings.

In the development of science, the so-called *scientific (r)evolutions* (M. Kokowski's terminology) happen. They are the result of both evolutionary and revolutionary processes, because "scientific evolutions" and "scientific revolutions" are complementary descriptions of the development of science (Kokowski 2015).

Appendix 2.

Letter from Władysław Natanson to Alfred Landé (14 November 1925)

1. Scan of the letter: pages 1–4.

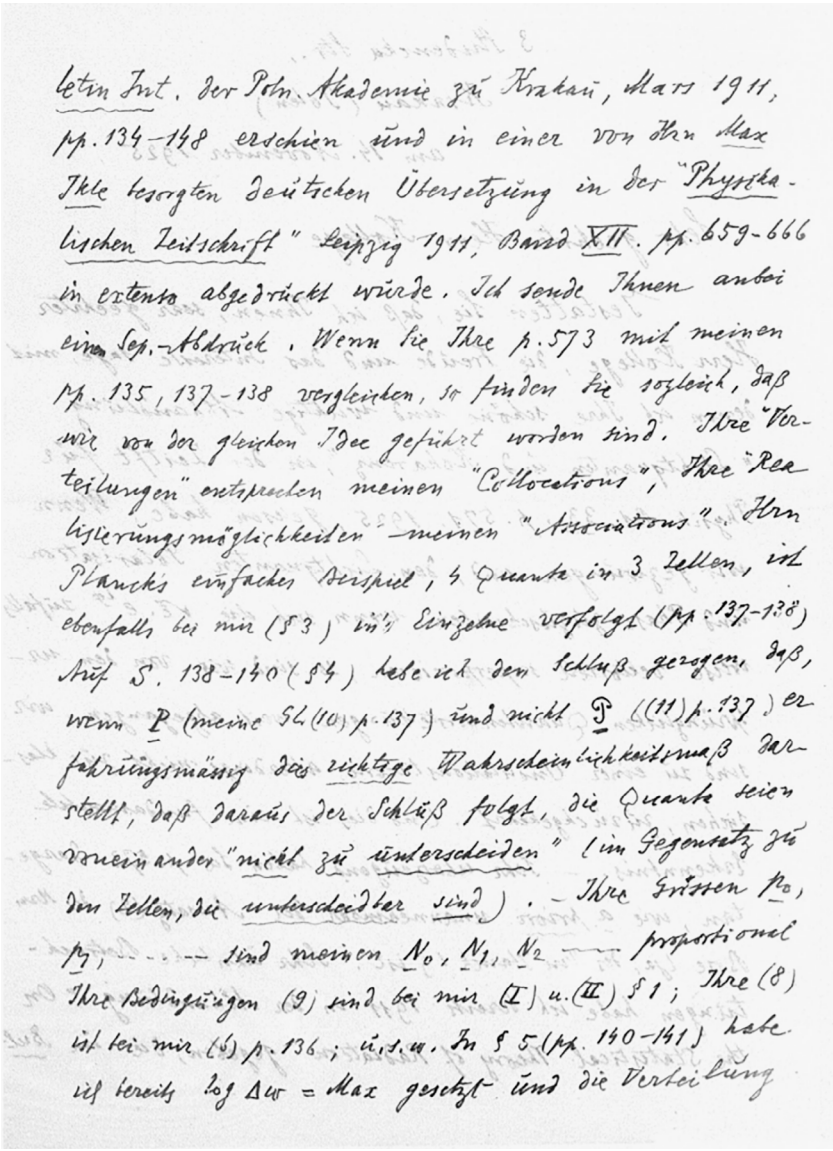


3 Studencka Str.,
Krakau (Polen)
am 14. November 1925

Sehr geehrter Herr Kollege

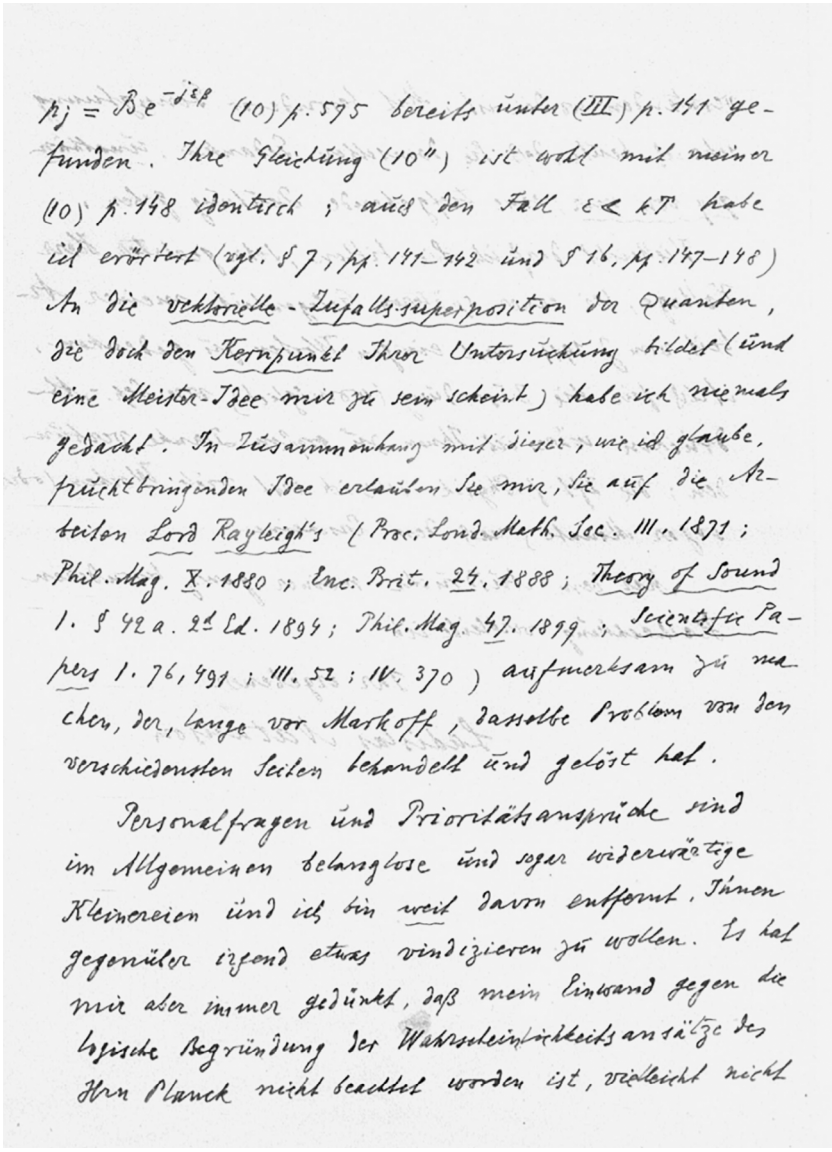
Gestatten Sie, daß ich Ihnen, sehr geehrter Herr Kollege, die Freude und das Interesse sage, mit denen ich Ihre schöne und wichtige Abhandlung "Lichtquanten und Kohärenz", in der Zeitft für Physik Bd. 33 . p. 571. 1925, gelesen habe. Wenn wir gezwungen sind, den Lichtquanten Polarisation und Phase zuzuschreiben, wenn sich die $\vec{V}.e^{i\varphi}$ zufällig. weise vektoriell superponieren, so sind wir von den ursprünglichen Quantenvorstellungen weit abgegangen, wir sind zu einer Undulationstheorie, allerdings nicht für klassischen, zurückgekehrt. Und dies ist eine fundamentale Erkenntnis. – Sehr überzeugend haben Sie p. 573 dargetan, wie a priori unannehmbar der Ansatz (2) des Hrn Bose (ja, des Hrn Planck!) ist. Sehr ähnliche Betrachtungen habe ich bereits 1911 in der Abhandlung "On the Statistical Theory of Radiation" gegeben, die im Bul-

Page 1. Source: © Archive for the History of Quantum Physics, 1898–1950. Volume reel 4: reel-frame 19, Call Number Mss.530.1.Ar2.



letzt Int. der Poln Akademie zu Krakau, Mars 1911, pp. 134–148 erschien und in einer von Hrn Max Ihle besorgten deutschen Übersetzung in der "Physikalischen Zeitschrift" Leipzig 1911, Band XII. pp. 659–666 in extenso abgedruckt wurde. Ich sende Ihnen anbei einen Sep.-Abdruck. Wenn Sie Ihre p. 573 mit meinen pp. 135, 137–138 vergleichen, so finden Sie sogleich, daß wir von der gleichen Idee geführt worden sind. Ihre "Verteilungen" entsprechen meinen "Collocations", Ihre "Realisierungsmöglichkeiten" meinen "Associations". Ihr Plancks einfaches Beispiel, 4 Quanta in 3 Zellen, ist ebenfalls bei mir (§ 3) im Einzelne verfolgt (pp. 137–138) Auf S. 138–140 (§ 4) habe ich den Schluß gezogen, daß, wenn P (meine Gl. (10) p. 137) und nicht P ((11) p. 137) erfahrungsmässig das richtige Wahrscheinlichkeitsmaß darstellt, daß daraus der Schluß folgt, die Quanta seien voneinander "nicht zu unterscheiden" (im Gegensatz zu den Zellen, die unterscheidbar sind). – Ihre Grössen p_0, p_1, ... sind meinen N_0, N_1, N_2 —— proportional Ihre Bedingungen (9) sind bei mir (I) u. (II) § 1; Ihre (8) ist bei mir (6) p. 136; u.s.w. In § 5 (pp. 140–141) habe ich bereits $\log \Delta w = \max$ gesetzt und die Verteilung

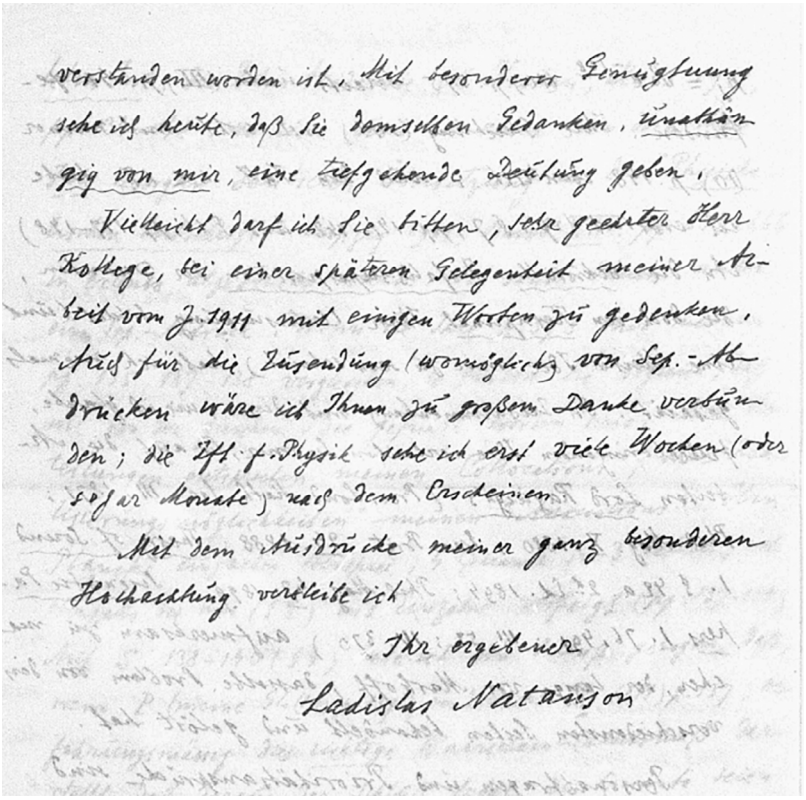
Page 2. Source: © Archive for the History of Quantum Physics, 1898–1950. Volume reel 4: reel-frame 19, Call Number Mss.530.1.Ar2.



$p_j = Be^{-\frac{\varepsilon_j}{kT}}$ (10) p. 595 bereits unter (III) p. 141 gefunden. Ihre Gleichung (10″) ist wohl mit meiner (10) p. 148 identisch; aufs den Fall $\varepsilon \ll kT$ habe ich erörtert (vgl. § 7, pp. 141–142 und § 16, pp. 147–148) An die vektorielle - Zufalls-superposition der Quanten, die doch den Kernpunkt Ihrer Untersuchung bildet (und eine Meister-Idee mir zu sein scheint) habe ich niemals gedacht. In Zusammenhang mit dieser, wie ich glaube, fruchtbringenden Idee erlauben Sie mir, Sie auf die Arbeiten Lord Rayleigh's (Proc. Lond. Math. Soc. III. 1871; Phil. Mag. X. 1880; Enc. Brit. 24. 1888; Theory of Sound I. § 42 a. 2te Ed. 1894; Phil. Mag. 47. 1899; Scientific Papers I. 76, 491; III. 52; IV. 370) aufmerksam zu machen, der, lange vor Markoff, dasselbe Problem von den verschiedensten Seiten behandelt und gelöst hat.

Personalfragen und Prioritätsansprüche sind im Allgemeinen belanglose und sogar widerwärtige Kleinereien und ich bin weit davon entfernt, Ihnen gegenüber irgend etwas vindizieren zu wollen. Es hat mir aber immer gedünkt, daß mein Einwand gegen die logische Begründung der Wahrscheinlichkeitsansätze des Herr Planck nicht beachtet worden ist, vielleicht nicht

Page 3. Source: © Archive for the History of Quantum Physics, 1898–1950. Volume reel 4: reel-frame 19, Call Number Mss.530.1.Ar2.



verstanden worden ist. Mit besonderer Genugtuung sehe ich heute, daß Sie denselben Gedanken, unabhängig von mir, eine tiefgehende Deutung geben.

Vielleicht darf ich Sie bitten, sehr geehrter Herr Kollege, bei einer späteren Gelegenheit meiner Arbeit vom J. 1911 mit einigen Worten zu gedenken. Stets für die Zusendung (womöglich von Sep.-Abdrücken wäre ich Ihnen zu großem Danke verbunden; die Zft. f. Physik sehe ich erst viele Wochen (oder sogar Monate) nach dem Erscheinen.

Mit dem Ausdrucke meiner ganz besonderen Hochachtung verbleibe ich

Ihr ergebener

Ladislas Natanson

Page 4. Source: © Archive for the History of Quantum Physics, 1898–1950. Volume reel 4: reel-frame 19, Call Number Mss.530.1.Ar2.

2. Text transcription.

3 Studencka Str.
Krakau (Polen)
am 14. November 1925

Seer geehrter Herr Kollege

Gestatten Sie, daß ich Ihnen, sehr geehrten Herr Kollege, die Freude und das Interesse sage, mit denen ich Ihre schöne und wichtige Abhandlung "Lichtquanten und Kohärenz", in der Zeit[schri]ft für Physik Bd. 33. p. 571. 1925, gelesen habe. Wenn wir gezwungen sind, den Lichtquanten Polarisation und Phase zuzuschreiben, wenn sich die $\sqrt{\varepsilon} \cdot e^{i\varphi}$ zufallsweise vektoriell superponieren, so sind wir von den

ursprünglichen Quantenvorstellungen weit abgegangen, wir sind zu einer Undulationstheorie, allerdings nicht zur klassichen, zurückgekehrt. Und dies ist eine fundamentale Erkenntnis. – Sehr überzeugend haben Sie p. 573 dargetan, wie a priori unannehmbar der Ansatz (2) des H[er]rn Bose (ja, des He[r]rn Planck!), ist. Sehr ähnliche Betrachtungen habe ich bereits 1911 in der Abhandlung "On the statistical Theory of Radiation" gegeben, die im *Bul-*

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letin Int. der Poln. Akademie zu Krakau [*Bulletin Int. der Akademie zu Krakau*], Mars 1911, pp. 134–148 erschien und in einer von H[er]rn Max Ihle besorgten deutschen Übersetzung in der "Physikalischen Zeitschrift" Leipzig 1911, Band XII, pp. 659–666 in extenso abgedruckt wurde. Ich sende Ihnen anbei einen Sep.– Abdruck. Wenn Sie Ihre p. 573 mit meinen pp. 135, 137–138 vergleichen, erfinden Sie sogleich, daß wir von der gleichen Idee geführt worden sind. Ihre "Verteilungen" entsprochen meinen "Collocations", Ihre "Realisierungsmöglichkeiten" – meinen "Associations". Herr Planck's einfaches Beispiel, 4 Quanta in 3 Zellen, ist ebenfalls bei mir (§ 3) ins Einzelne verfolgf (pp. 137–138). Auf S. 138–140 (§ 4) habe ich den Schluß gezogen, daß, wenn P (meine SL (10) p. 137) und nicht P ((11) p. 137) erfahrungsmässig das wichtige Wahrscheinlichkeitsmaß darstellt, daß daraus der Schluß folgt, die Quanta seien voneinander "nicht zu unterscheiden" (in Gegensatz zu den Zellen, die unterscheidbar sind). – Ihre Grössen p_0, p_1, … sind meinen N_0, N_1, N_2, … proportional Ihre Bedingungen (9) sind bei mir (I) u. (II) §1; Ihre (8) ist bei mir (6) p. 136; u.s.w. In § 5 (pp. 140–141) habe ich bereits log Δw = Max gesetzt und die Verteilung

p. 3

$p_j = B\, e^{-jr\beta}$ (10) p. 575 bereits unter (III) p. 141 gefunden. Ihre Gleichung (10") ist wohl mit meiner (10) p. 148 identisch; auf den Fall $\varepsilon \ll kT$ habe ich erörtert (vgl. § 7, pp. 141–142 und § 16, pp. 147–148). An die vektorielle - Zufalls-superposition der Quanten, die doch den Kernpunkt Ihrer Untersuchung bildet (und eine Meister-Idee mir zu sein scheint) habe ich niemals gedacht. In Zusammenhang mit dieser, wie ich glaube, fruchtbringenden Idee erlauben Sie mir, Sie auf die Arbeiten Lord Rayleigh's (*Proc. Lond. Math. Soc.* III, 1871; *Phil. Mag.* X 1880; *Enc.*

Brit. 24.1888; *Theory of Sounds* 1. § 42a. 2nd ed. 1884; *Phil. Mag.* 47. 1899; *Scientific Papers* I. 76, 491; III. 52; IV. 370) aufmerksam zu machen, der, lange vor Markoff, dasselbe Problem von den verschiedensten Seiten behandelt und gelöst hat.

Personalfragen und Prioritätsansprüche sind im Allgemeinen belanglose und sogar widerwärtige Kleinereien und ich bin weit davon entfernt. Ihnen gegenüber irgendetwas vindizieren zu wollen. Es hat mir aber immer gedünkt, daß mein Einwand gegen die logische Begründung der Wahrscheinlichkeitsansätze des H[er]rn Planck nicht beachtet worden ist, vielleicht nicht

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verstanden worden ist. Mit besonderer Genugtuung sehe ich heute, daß Sie demselben Gedanken, unabhängig von mir, eine tiefgehende Deutung geben.

Vielleicht darf ich Sie bitten, sehr geehrter Herr Kollege, bei einer späteren Gelegenheit meiner Arbeit von J. 1911 mit einigen Worten zu gedenken. Auch für die Zusendung (womöglich) von Sep. – Abdrucken wäre ich Ihnen zu großen Danke verbunden; die *Z[eitschri]ft f. Physik* sehe ich erst viele Wochen (oder sogar Monate) nach dem Erscheinen.

Mit dem Ausdrucke meiner ganz besonderen Hochachtung verbleibe ich

Ihr ergebener
Ladislas Natanson

3. English translation

3 Studencka Street
Kraków (Poland)
on November 14, 1925

Dear Colleague,

Allow me to say you, dear colleague, the joy and interest with which I read your beautiful and important treatise "Lichtquanten und Kohärenz", in the *Zeit[schri]ft für Physik* vol. 33 (1925), p. 571. If we are forced to attribute a polarization and a phase to the quantum of light, then the $\sqrt{\varepsilon}\cdot e^{i\varphi}$ happens to be random.

We superpose vectors, so we are far from the original quantum ideas, we have returned to an undulation theory, but not to the classical. And this is a fundamental realization. – You have showed very convincing on p. 573 how a priori unacceptable is the distribution (2) of Mr. Bose (and yes, of Mr. Planck!). Already in 1911 I gave very similar considerations in the treatise "On the Statistical Theory of Radiation", which is published in *Bulletin Int. the Poln. Academy of Krakow* [*Bulletin Int. Academy of Krakow*], March 1911, pp. 134–148 appeared and in a German translation procured by Mr. Max Ihle in the *Physikalischen Zeitschrift* Leipzig 1911, volume XII, pp. 659–666 in extenso. I send you a Sept[ember]. imprint. If you compare your p. 573 with my pp. 135, 137–138, you will find at once that we have been led by the same idea. Your "distributions" correspond to my "collocations", your "realization of possibilities" – my "associations". Mr. Planck's simple example, 4 quanta in 3 cells, is also closely followed by me (§ 3) (pp. 137–138). On pp. 138–140 (§4), I have concluded that if $\underline{P}$ (my SL (10) p. 137) and not $\underline{P}$ ((11) p. 137) is the most important measure of the probability, that out of it the conclusion follows that the Quanta are "indistinguishable" from each other (in contrast to the cells that are distinguishable). – Your magnitudes p_0, p_1, ... are proportional to my N_0, N_1, N_2, ... Your conditions (9) are with me (I) and (II) §1; Your (8) is with me (6) p. 136; etc. In § 5 (pp. 140–141) I have already set $\log \Delta w = \text{Max}$ and the distribution $p_j = B\, e^{-j\varepsilon\beta}$ (10) p. 575 already under (III) p. 141 found. Your equation (10") is also identical with my (10) on p. 148; I have discussed the case of $\varepsilon \ll kT$ (see § 7, pp. 141–142 and § 16, pp. 147–148). I have never thought of the random vectorial superposition of quanta, which is the core of your investigation (and seems to me to be a master idea). In connection with this, as I believe, fruitful idea, permit me to refer you to the works of Lord Rayleigh (*Proc. Lond. Math. Soc.* III, 1871; *Phil. Mag.* X, 1880; *Enc. Brit.* 24, 1888; *Theory of Sounds* I, § 42a, 2nd ed., 1884; *Phil., Mag.* 47, 1899; *Scientific Papers* I, pp. 76, 491; III, p. 52; IV, p. 370), who, long before Markoff, dealt with and solved the same problem from various sides.

Personnel issues and priority claims are generally trivial and even obnoxious, and I'm a long way from that. I want to assure you of something. Well, it always seemed to me that my objection to the logical rationale of Mr. Planck's probabilistic approach was not taken into account [because] it may not have been understood. Today I am

especially glad that, independently of me, you provide a profound interpretation of the same ideas.

Perhaps, may I ask you, dear colleague, to mention in a later opportunity my work of 1911 in a few words. Also I would also be very grateful if you could send me (if possible) imprints from the September issue of the *Z[eitschri]ft f[ür]. Physik* – I see this journal only many weeks (or even months) after the publication.

With the expression of my very special respect, I remain
Yours faithfully

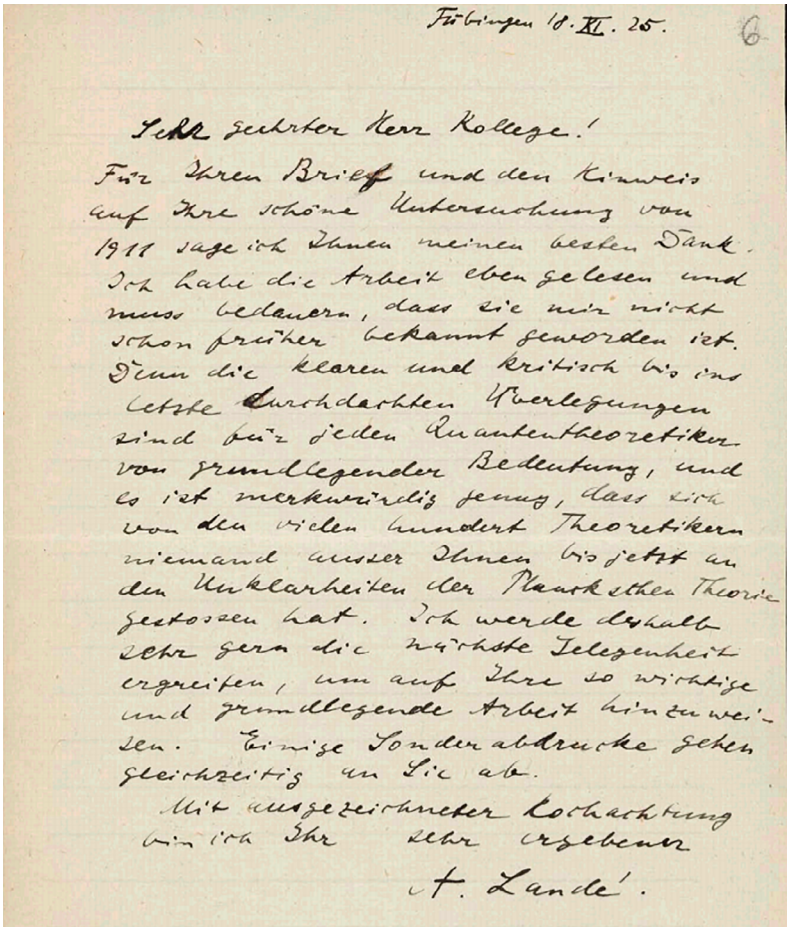
Ladislas Natanson[39]

[39] For a Polish translation see Kokowski 2021, Dodatek 2.

Appendix 3.
Letter from Alfred Landé to Władysław Natanson (Tübingen, November 18, 1925)

1. Scan of the letter

In: Korespondencja Władysława Natansona z lat 1884–1937. T. 10, Laar – Łukasiewicz [Rękopis]. Available online: https://jbc.bj.uj.edu.pl/publication/610285, folium 6.



Tübingen 18. XI. 25.

6

Sehr geehrter Herr Kollege!

Für Ihren Brief und den Hinweis auf Ihre schöne Untersuchung von 1911 sage ich Ihnen meinen besten Dank. Ich habe die Arbeit eben gelesen und muss bedauern, dass sie mir nicht schon früher bekannt geworden ist. Denn die klaren und kritisch bis ins letzte durchdachten Überlegungen sind für jeden Quantentheoretiker von grundlegender Bedeutung, und es ist merkwürdig genug, dass sich von den vielen hundert Theoretikern niemand ausser Ihnen bis jetzt an den Unklarheiten der Planck'schen Theorie gestossen hat. Ich werde deshalb sehr gern die nächste Gelegenheit ergreifen, um auf Ihre so wichtige und grundlegende Arbeit hinzuweisen. Einige Sonderabdrucke gehen gleichzeitig an Sie ab.

Mit ausgezeichneter Hochachtung bin ich Ihr sehr ergebener

A. Landé.

Source: © Biblioteka Jagiellońska. „Korespondencja Władysława Natansona z lat 1884–1937", T. 10, Laar – Łukasiewicz [Rękopis]. Available online: https://jbc.bj.uj.edu.pl/publication/610285, folium 6.

2. Text transcription

Tübingen 18. XI.25.

Sehr geehrter Herr Kollege!

Für Ihren Brief und den Hinweis auf Ihre schöne Untersuchung von 1911 sage ich Ihnen meinen besten Dank. Ich habe die Arbeit eben gelesen und muss bedauern, das sie mir nicht schon früher bekannt geworden ist. Dein die klaren und kritish bis ins letzte durchdachten Überlegungen sind für jeden Quantentheoretiker von grundlegenter Bedeutung, und es ist merkwürdig genug, dass sich den vielen hundert Theoretikern niemand ausser Ihnen bis jetzt an den Unklarheiten der Planckschen Theorie gestossen hat. Ich werde deshalb sehr gern die nächste Gelegenheit ergreifen, um auf Ihre so wichtige und grundlegende Arbeit hinzuweisen. Einige Sonderabdrucke gehen gleichzeitig an Sie ab.

Mit ausgezeichneter Hochachtung bin ich Ihr sehr ergebener.

A. Landé

3. English translation

Tübingen, 18 November 1925

Dear Colleague!

Thank you very much for your letter and the reference to your beautiful investigation of 1911. I have just read the work and regret that it had not become known to me earlier. Your clear and critical, down to the last one, thoughtful considerations are of fundamental importance to any quantum theorist, and it is curious enough that none of many hundreds of theorists apart from you, have grasped the uncertainties of Planck's theory. Therefore, I am eager to seize the next opportunity to point out your important and fundamental work. Some offprints will be sent at the same time to you.

Respectfully yours,

A. Landé

Appendix 4.
The spectrum of Władysław Natanson's positions on the ontological status of energy quanta

Natanson spoke on the title topic in 1911–1925 and formulated various positions. This is evidenced by his comments presented in his publications: Natanson 1911a; 1911b (1st version) / 1912a (= offprint 1911b); 1911c; 1912a; 1912b; 1913; 1923; 1924b (= version II 1911b).

In his best-known articles 1911a / 1911c and also 1912b, Natanson omitted this issue. It could be concluded from this that he was treating energy elements / units of energy as fictitious calculations, since he mentioned that Boltzmann treated them as such in 1872.

However, in the paper entitled "On radiation", delivered on July 19, 1911 during the 11th Congress of Polish Doctors and Naturalists in Kraków (July 18–22, 1911), Natanson (1911b / 1912a) expressed an unambiguous view that energy quanta exist in reality, because physics of gas phenomena and light radiation (specifically the existence of the correspondence principle linking Boltzmann-Planck-Natanson statistics with Maxwell-Boltzmann statistics) speaks for it; however, energy quanta are not immutable atoms:

Natanson 1911b, pp. 158–159

> We sometimes hear or read statements that the hypothesis on finite energy units is useful because it allows us to calculate the probabilities of various distributions of energy and thus indicates the most likely one among them. If it was established solely for this purpose, the hypothesis of units would, eventually, be merely a result of the calculation. However, it seems to me that by no means it is one. If we suppose that a unit of energy becomes ever *smaller* and tends to zero, then the most likely energy distribution does not become vague or indefinite at all, but tends to shift (at least in essence) into the classic Maxwellian energy distribution known from the Gas Theory. In his Radiation Theory P1anck departed from Maxwell's type of distribution and shifted to an almost opposite extreme. In order to understand this, we should remember that

when the *quanta* are small, then their number in a given system must therefore be large.

[...] Let us now take the position of the hypothesis of elementary units of energy, and let us say in general terms: to arrive at Maxwell's way of distributing energy in the Kinetic Theory of Gases we must suppose that the ratio of the present units to the number of molecules is enormous or even that, practically speaking, it is infinitely large. On the contrary, in the Radiation Theory the ratio n / N is a small fraction; formula (1) of article X [this is Planck's radiation law of energy distribution] follows this assumption. In Gas Theory, then, we consider the case of a system that is *richly* endowed with energy; in the Radiation Theory, on the contrary, we see the consequences when the system is sparse, poor energized. We have here two extreme and directly opposite, very special cases of a much more general but also more complex regularity. This regularity, which we will have to learn to use, is undoubtedly the foundation of the future Kinetic Theory of all states of matter aggregation. Already today we can see the beginnings of the Kinetic Theory of Solids, which Einstein, with ordinary boldness and with considerable success, recently began to build according to analogous outlines. We can briefly say that this Theory is dominated by what lies at the bottom of Planck's law (translation – M. Kokowski).[40]

There are elementary *quanta* or units of energy; this is what we are taught, in the light of the Probability Calculus, by cardinal facts in the Field of Radiation. And this result immediately captures all our thinking (translation – M. Kokowski).[41]

But what is an 'atom of energy'? Is it possible to understand the connectivity, coherence, indivisibility of a cer-

[40] For the Polish original see Natanson 1911b, pp. 158–159 & Kokowski 2021, Dodatek 3.

[41] For the Polish original see Natanson 1911b, p. 157 & Kokowski 2021, Dodatek 3.

tain amount of energy? Similar questions arise spontaneously. If we were to try to express, in the most general formulation, the essential *content* of the value that we have been gained by Planck's theory, could we not say that in the fundamental facts of radiation Planck noticed the existence of a certain *discontinuity* in Nature. But *where* is this discontinuity?

We can imagine it directly in the emission and absorption of radiation by the ultimate particles of matter or, as Planck has shown recently, we can transfer it to the emission only. Returning to Newton, we might suppose that the discontinuity is inherent to the spatial nature of radiation, as some scientists have tried to reason; or maybe in the structure of universal ether. Given that we encounter a discontinuity in a concept, namely the concept of energy, it is easy to see that we can extend this discontinuity, as it were, onto various orders of thought. Perhaps we will place it finally in the concept of time or in the concepts of pure geometry but we cannot get rid of it by any means.

If so, can a *quantum* of energy be understood dynamically or electromagnetically at all? The Dynamics and Electromagnetic Theory, at least in its ordinary present form, operates with concepts that are always continuous, and both, in their classical form, lead to the Principle of Energy Equipartition, and this, as we know, is a contradiction of today's proven formula of Radiation. We get tangled up in contradictions when we try to imagine *quanta* of energy as shipments, some charges or bullets of energy, running through ordinary space and time according to ordinary laws that look for continuity in everything. We could have foreseen this failure. Let us say only briefly that a continuous model of discontinuous energy units is *impossible*. But consider this: do we really need it? The concept of energy units becomes an element of our understanding of Nature. If it turns out, as it may be supposed, to be its indispensable factor, then it must penetrate all our thinking and transform everything that

> is not compatible or consistent with it today (translation – M. Kokowski).[42]

Despite such unambiguous statements, Natanson doubted whether he was right about understanding the quanta. In 1913, commenting in the dissertation entitled "Principles of Theory of Radiation" ("Principes de la Théorie du Rayonnement") the choice of Boltzmann's probability measure P instead of P, Natanson quoted a critical remark by a Polish physicist Kamil Kraft (1873–1945):

> What lies at the bottom of this fact? What can one learn from it? We are still a long way from having answers to these and other questions. One might suppose (Dr. K. Kraft made this remark) that the physical meaning of the concept P and the failure of P are indicative of the existence of molecular discontinuities in nature and the non-existence of real 'quantas' or discontinuities of energy. But we should confess that this guideline is still unclear and weak today.[43]

Ten years later, on April 4, 1923, in a speech at the inaugural session of the first Congress of Polish Physicists and Chemists in Warsaw, bearing in mind a) *the non-existence of a unified theory of optical phenomena* and b) the existence of a contradiction in the descriptions of optical phenomena offered by wave optics and the (new) quantum optics, Natanson saw serious limitations of the so far postulated hypothetical entities of theories:

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[42] For the Polish original see Natanson 1911b, pp. 159–160 & Kokowski 2021, Dodatek 3.

[43] For the Polish original see Natanson 1913, pp. 60–61 & Kokowski 2021, Dodatek 3.

From Kamil Kraft's letter to Natanson of August 20, 1912, we know that almost three weeks earlier (i.e. at the beginning of August 1912), Kraft received from Natanson his dissertation "Zasady teoryi promieniowania" ("Principles of the theory of radiation"), published only in 1913 (Natanson 1913) and he highly appreciated this study without reading it yet – see Kraft 1912 (*archival document*). Kamil Kraft is another physicist who knew and appreciated Natanson's achievements in the field of Planck radiation theory. I overlooked him in my previous article – see Kokowski 2019, pp. 368–381. Unfortunately, there are no other letters from Kraft in the existing Natanson correspondence concerning this issue.

> *So let us not believe in the existence of correct and simple waves; let us also refuse to trust in the objective existence of quantum atoms.* The incompatibility of undulation and quantum Optics only testifies to the fact that both disciplines, having picked up a thin thread of similarity in the phenomena, extended it, each its own, beyond their proper scope. They assumed identity out of analogy, they took the subtle metaphor literally, naively and crudely (italic, translation – M.K.).[44]

In 1924, Natanson, in the study entitled "On radiation", which is the second version of an article from 1911 (Natanson 1911b), presented a final position on the ontological status of energy quanta. According to that, energy quanta do exist, but it still creates conceptual difficulties on the grounds of physics.

Natanson 1924b, pp. 142–143:

> Boltzmann made a very important theorem: there is a relationship between the probability of energy distribution in a system and its entropy. But this relationship satisfies the probability of distribution, computed according to Boltzmann's expression. Otherwise calculated probability does not satisfy it. We must invoke this fact in order to justify a posteriori why we follow Boltzmann, Planck, in determining the probability of distribution. What lies at the bottom of this fact? What can you learn from it? Despite long discussions, we do not have an unquestionable answer to these and similar questions. And while Boltzmann's theorem plays an increasingly important role in the most

[44] For the Polish original see Natanson 1923, p. 8 & Kokowski 2021, Dodatek 3.

In the terminology developed by the author of the article *the hypothetical-deductive method of correspondence-oriented thinking*, the hypothetical beings postulated in the context of theory are actually quasi-beings – see Kokowski 1996. In the case under consideration, Natanson saw a striking contradiction between the quasi-entities postulated by wave optics and the new quantum optics, and saw the need to search for a more general theory of optical phenomena with new quasi-entities that would avoid the contradictions of these theories, while preserving the already discovered partial truths – laws of phenomena consistent with the predictions of the hitherto theories; on partial truths in the mathematico-physical sciences, see Costa, French 2003.

difficult research today, we are reaffirmed, quite bitterly, in the observation that great generalizations of science, while they are fruitful and beneficial, tend to be the most obscure (translation – M. Kokowski).[45]

Natanson 1924b, pp. 143–144:

Following this path, Planck was led to the famous formula that bears his name and which, as far as we know, agrees well with the results of measurements within the limits of unavoidable perception errors. We are not going to quote Planck's formula here; rather, we will highlight its most important and immediate conclusion. Let us denote by an elementary quantity or a unit of energy, with a certain selected frequency; letters k and T let us keep their previous purpose. Planck's formula then shows us that the mean value of the energy of one vibrator (or a fundamental vibration) is not kT, as required by the principle of energy equipartition; the ratio of this mean to kT does not equal 1, but depends again on the ratio of e to kT. Although the mean value of energy of the vibrator or the vibration would tend to reach the limit value of kT if we made e disappear, but the reality does not allow this to be assumed; for as it tends to zero, then Planck's formula at the limit transforms into Rayleigh's law, and we know for certain that this law denies real validity and truth. We must therefore leave e finite in Planck's formula, other than zero. *There are finite quanta or units of energy; the reality of radiation teaches us this in the light of calculations* (italics, translation – M. Kokowski).[46]

Natanson 1924b, pp. 144–145:

By exploring the conditions of the equilibrium of radiation, *have we now discovered the quanta or energy atoms*? Should we really imagine coherent and cumulative, indivisible, unchanging

[45] For the Polish original see Natanson 1924b, pp. 142–143 & Kokowski 2021, Dodatek 3.

[46] For the Polish original see Natanson 1924b, pp. 143–144 & Kokowski 2021, Dodatek 3.

units of radiant energy? Huge difficulties await us here. Chemical atoms, at least in yesterday's classical chemistry, are unchangeable; an H or O or Cl atom is the simplest element, inviolable in (ordinary) chemical reactions. The electron is likewise the simplest constant unit of charge in electromagnetic phenomena. The unit e of energy is, on the contrary, something complex, dependent and variable. According to Planck, e depends on the frequency of the considered elementary radiation. In order to agree with the general thermodynamic laws of equilibrium, Planck assumed that the unit e is directly proportional to the frequency of vibration of the radiation or inversely proportional to its wavelength. Depending on the nature, on the pulse of the radiation, the variable quantity e cannot be a permanent, eternal atom, it cannot be the primary element that the world of phenomena is made of. But if we multiply the quantity e by the period τ of the radiation vibrations, we get the invariant eτ. The product of energy by time is called action in dynamics; the important claim of this science is called the principle of least action. In the modern quantum theory we therefore claim today that in physical phenomena, or at least in radiation, there exists a certain constant and universal quantity, a certain standard of action; this norm eτ or shorter h, as it was assumed to be written after Planck, is

$$h = 6.55 \times 10^{-27} \times \text{erg} \times \text{sec.}$$

If we knew exactly the form of the laws governing the tiniest changes in Nature, we would probably know what h means; in this respect, Bohr's theory, which also refers to the norm h, gives rise to some hope. We are temporarily satisfied with the (undoubtedly abstract) assumption that the universal constant h represents the minimum possible action; from units, from h atoms we raise a variety of movements and changes in the ultimate fragments of the world available to us[47] (translation – M. Kokowski).

[47] For the Polish original see Natanson 1924b, pp. 144–145 & Kokowski 2021, Dodatek 3.

Natanson 1924b, pp. 149–151

If we were to try in the most general way to express the essential content of the achievement that we have been enriched with by the quant theory, we would probably have to say that in the fundamental facts of radiation we have noticed the existence of a certain discontinuity in Nature. We then found it in a variety of other physical phenomena, which we have mentioned very briefly or not at all in this elusive sketch. But where is this discontinuity? Planck originally envisioned it in the processes of emission and absorption by particles of matter; later he tried to limit it to just emissions. As we have seen, Niels Bohr gave Planck's assumption a concrete and clear form, at the same time seizing an enormous range of spectral phenomena under the rule of theory. Returning to some extent to the optical images of Newton, Einstein, Stark, J. J. Thomson and other scientists wanted to see a discontinuity in the fibrous structure of radiation, in its being spatially woven, as it were. In its simplest form, the discontinuity, as we know, takes the concept of a dynamic action, which in Minkowski's four-dimensional world becomes a fundamental concept. Yet, we can apply it as we wish in various orders of thought. We will put it, perhaps, in geometric concepts; we will declare a vacuum for a cracked, discontinuous creature. In all likelihood, we will agree someday that time is atomistic, as Poincaré had already semi-ironically recommended. However, by no means can we get rid of the discontinuities. We have suspected that there is some discontinuity somewhere deep in the world of events; and it is a fresh and strong thought that can infiltrate our whole understanding of Nature, one that can transform everything that does not agree with it. Today, however, it encounters insurmountable difficulties in optics. The wealth of experience accumulated in optics over the course of several centuries, rich and accurate and reliable, stands in blatant contradiction to any hypothesis of light packets, charges or projectiles. In view of the precision of optical phenomena, the quantum thought seems not only powerless or superfluous, but also incompatible with itself. Only a finite sequence of radiation disturbances in a vacuum can result from an emission of a finite amount of energy; only a finite sequence can trigger absorption of a finite amount of energy. Meanwhile, the usual concepts of wavelength and the frequency or period of oscillation do not apply at all to a finite, breaking sequence of waves; they are concepts

marked and clear in an infinite terms only, in an everlasting, infinitely extensive sequence, the amplitude of which is absolutely constant. Thus, the concepts of wavelength, frequency and period of oscillation do not have definitions in quantum theories. When we mention n, λ or τ in these theories, we threaten the Nature with a tool that we shattered ourselves. But let's trust in the future. Puzzles that are incomprehensible today will be most likely explained by the radiant tomorrow. In the still prevailing darkness, touched by a grand idea, albeit so far unclear, an unexpected and branched weave of truth springs from the unfathomed field of reality (translation – M. Kokowski).[48]

Appendix 5.
Selected aspects of a sketchy discussion with the theses of some critics of Kastler and Natanson

Artur Kastler (1983, p. 616), not knowing the content of Natanson's letter to Landé, applied the same Natanson's reinterpretation strategy: he identified Natanson's "energy units" with particles (e.g. photons) and Natanson's "ultimate particles" or "receptacles of energy" with "cells in phase space".

However, such a reinterpretation of Kastler was negatively assessed by John Stachel (2000/2002, pp. 438–439) as inaccurate, because it is anachronistic: Natanson considered the distribution of discrete elements or units of energy in material entities and not light quanta in phase cells, because he identified energy carriers as "ultimate particles constituting the matter. [...] capable only of absorbing, containing and emitting amounts of energy, which are multiplications of these finite and determinate units" (Natanson 1911a, p. 134), analogous to how Planck considered charged oscillators in equilibrium with the radiation field at a certain temperature. This, according to Stachel, had nothing to do with the distribution of particles (radiation or some other kind) in the phase space cells, and therefore there is no reason to question Einstein's lack of response to Natanson's article.

On the other hand, Silvio Bergia (1987, p. 344) and Jagdish Mehra and Helmut Rechenberg (1982/2001, p. 559) believed that Natanson

[48] For the Polish original see Natanson 1924b, pp. 149–151 & Kokowski 2021, Dodatek 3.

did not provide any physical explanation for the derivation of Planck's distribution, and that was the fundamental deficiency in Natanson's interpretation.

In claiming this, the critics of Natanson and Kastler overlooked a key aspect of the arguments, as these interpretations are based on a translation of the language of one model into the language of another model. In the article from 1911, Natanson indicated *corresponding* elements of Planck's model and Natanson's model, and in the letter to Landé of 1925, such elements of Bose's model and Natanson's model, which later – in 1983 – was also done by Kastler, and earlier – in 1925 – also by Landé, who compared Bose's model and Planck's model.

Daniela Monaldi also continued the type of criticism proclaimed by John Stachel: "[…] Natanson's language can easily sound like an anticipation of today's common characterization of quantum statistics as a statistics of indistinguishable particles, but the resemblance is only retrospective. Natanson was simply re-discovering the combinatorial difference already stressed by Planck. Natanson did not draw, implicitly or explicitly, the conclusion that counting energy elements as «undistinguishably alike» amounted to a new statistics. He kept within the possibilities of the current statistical style of reasoning. For him, as for Planck, it was expected and unremarkable that energy elements should be counted differently from receptacles, since the energy elements did not belong to the class of statistical objects. Comparing radiation and gas, he concluded that they differed only because the receptacles of radiation were poorly endowed with energy while energy was abundantly bestowed upon the gas receptacles" (Monaldi 2019, p. 320; see also Natanson 1911a, p. 142; Stachel 2000; Monaldi 2009, pp. 387–388).

I do not agree with this interpretation of Daniela Monaldi:

1) Natanson's credit was not only in rediscovering the combinatorial difference emphasized by Planck, but in pointing out Planck's implicit assumptions about the indistinguishable elements of energy, which led to an explanation of the statistical basis of Planck's radiation theory, and thus we are entitled to speak of Boltzmann-Planck-Natanson statistics (Kokowski's term).

2) Natanson did not argue at all that the only difference between radiation and gas is very low or very high energy content, but that the magnitude of the n / N parameter (where n is the number of energy units and N is the number of vibrations, vibrators, particles, objects

in general), decides the type of statistics: when it is finite, we have Boltzmann-Planck-Natanson statistics, and when it is infinite this statistics is in the form of Maxwell-Boltzmann statistics. Consequently, Natanson indicated the correspondence parameter of the new theory – the Boltzmann-Planck-Natanson statistics (in Kokowski's terminology), the limiting case of which is the Maxwell-Boltzmann statistics.

Years later, this property of Einstein-Bose statistics was also recognized by Dirac in his fundamental monograph on quantum mechanics (1930, §68 Discussions of Einstein-Bose Assemblies, pp. 223–225). Similarly, Ehrenfest and Uhlenbeck (1927a), following Schrödinger, Fermi, Heisenberg and Dirac (1926, pp. 662, 667–673), presented a quantum (wave-mechanical) interpretation of the classical Maxwell-Boltzmann statistics (called by them the Boltzmann statistics). There is nothing surprising in this, as the quantum approach to classical statistics and quantum statistics inherit the achievements of earlier approaches to this issue, including the achievements of Natanson (these are specific "archaeological" layers of the theory).[49]

It should be emphasized that Natanson spoke extensively and clearly on these topics in his seven works on radiation statistics and related issues: Natanson 1911a (English version); 1911b (pp. 144–160); 1911c

[49] According to Alexander Bach (1997, pp. 138–139), Natanson revealed the existence of combinatorial indistinguishability of particles in opposition to the so-called physical indistinguishability of particles (defined by the idea of invariability of configuration probabilities with respect to permutation established in quantum mechanics and related to the problem of the "N!" factor in the ideal gas entropy formula). According to Natanson, the combinatorial indistinguishability of hypothetical entities is a consequence of the uniform probability distribution of the occupation number, while in quantum mechanics it is the result of the invariability of the configuration probabilities with respect to permutations; the uniform probability distribution is merely a trivial case of a symmetric measure of probability; the Natansonian understanding of indistinguishability was immediately accepted by Max Planck (1912/1914) and adopted in physics textbooks, and later became the basis of classical probability theory and combinatorics.

As for the so-called physical indistinguishability of particles, see also Hanle 1977; Darrigol 1988; 1991, Kunz 1989; van Fraassen 1998; French, Krause 2006; Dieks 2014; 2020; Swendsen 2015; Saunders 2020.

I will also add, which has been overlooked so far, that Natanson himself (1913, p. 58, fn. 2) referred in the context of the combinatorial indistinguishability of particles to the understanding of the probability of elementary events by Henri Poincaré (1896, §§ 1–10) and Władysław Gosiewski (1906, chap. 1).

(pp. 659–666); 1912a; 1912b; 1913 (this is Natanson's most important work on the theory of radiation); 1924b (pp. 125–151) – details see Kokowski 2019, pp. 336–343. Moreover, Alexander Bach (1990a, p. 24) emphasized the clarity of Natanson's approach (based on Boltzmann's approach to derive "Bose-Einstein" statistics), known to him from the German version of Natanson's article (1911c).

Additionally, Simon Saunders criticized Natanson's approach, referring to Daniela Monaldi's interpretation, but without quoting her in this context:

Natanson introduced needed clarity. He set out some of the combinatorial expressions of §2.5, including the identity of (18) and (19) (but so had Boltzmann). He gave a derivation, not of the Planck distribution, but of the Boltzmann probability distribution, Eq. (29) precisely following Boltzmann! His subsequent discussion of the difference between systems 'abundantly bestowed' with energy units, and those 'poorly endowed', missing as he did the coarse graining into frequency intervals, was flawed. He made no comment on statistical independence, or the lack of it. There may be no great puzzle here as to why his essay received so little commentary" (Saunders 2020, p. 49).

As in the case of Daniela Monaldi's interpretation, I disagree with Simon Saunders's interpretation, in particular with the statements that Natanson did not derive Planck's law, i.e. the Boltzmann-Planck-Natanson distribution / statistics (in Kokowski's terminology), but only the Boltzmann distribution (*because after all, Natanson alluded to both Boltzmann and Planck*); he did not comment on either statistical independence or the lack of it (*because it was Natanson who first drew attention to the hidden assumption of Planck's theory, i.e. the indistinguishability of energy elements*) and that he carried out an erroneous analysis of systems "abundantly endowed" with energy units and "poorly equipped" with these units (*this is not true*; my criticisms of Daniela Monaldi's interpretation apply here as well, see above).

I also refer here to specialist studies by Alexander Bach (Bach, Blank, Francke 1985; Bach 1985; 1987; 1988a; 1988b; 1990a; 1990b; 1990c; 1991; 1996, 1997, pp. 137–138), who explains in detail the intricacies of Boltzmann's approach in deriving Bose-Einstein statistics. I also share his thesis:

Conceptually, Bose derivation is identical to those of Lorentz and Natanson (Mehra, Rechenberg 1982, pp. 562–565). According to a fa-

miliar hypothesis (Pais 1982, pp. 427[–428]; Delbrück 1980; Darrigol 1986 [p. 212]), Bose arrived at his statistics erroneously,[50] by assuming a multinomial distribution for the occupancy numbers instead of a multinomial distribution for the occupation numbers. This hypothesis is misleading, however, because the occupancy numbers are subject to two constrains, whereas the occupation numbers are subject to only one constraint (Bach 1990a, s. 28).[51]

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[50] Daniela Monaldi also claims that Bose derived Planck's law by making a happy mistake. "Bose gathered that it was «now easy to calculate the thermodynamic probability of a (macroscopically defined) state»" [Bose 1924a, on 180]. There only remained to calculate the number of different ways in which the quanta in a given frequency interval could be distributed over the available cells. It was clearly not Bose's intention to introduce anything new in this part of the derivation. He admitted in a later interview that he had not been aware of doing anything different from what Boltzmann would have done. [Pais 1979, on 893]. In fact, Bose did use Boltzmann's formula for the probability of an energy distribution of an assembly of equal particles, but in applying it he inadvertently replaced the numbers of molecules having energies 0, e, 2e, 3e, … with the number of phase-space cells containing 0, 1, 2, 3, … light quanta. The replacement happily resulted in an energy density formula equivalent to Planck's law, and Bose regarded the operation a success" (Monaldi 2019, p. 324).

However, we are not dealing here with a mistake at all: the standard strategy of re-interpreting the physico-mathematical model according to the hypothetico-deductive method of correspondence-oriented thinking (this is the supra-historical style of mathematico-physical sciences) is manifested here. In this particular case, the strategy of analogical thinking was used: corresponding classes of hypothetical entities of the theory were indicated. On the role of analogy in mathematico-physical sciences, see Bailer-Jones 2009, pp. 46–80.

[51] See fn. 21.

einsteinpapers.press.princeton.edu/vol14-doc/779; English translation in: Einstein 2015b, Doc. 446, pp. 438–439. Available online: https://einsteinpapers.press.princeton.edu/vol14-trans/468.

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